We have introduced SSS, which says that a rp is SSS if all of its nth-order distributions do not depend on where the time origin is.

A weaker form of stationarity is often assumed instead

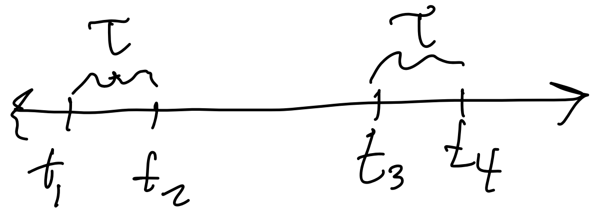
② (second type of stationarity)

A rp $X(t)$ is wide-sense stationary (WSS) if

- $\mu_X(t) = \mu_X$ does not depend on t, and
- $R_{XX}(t_1, t_2) = E[X(t_1) X(t_2)] = R_X(\tau)$ for some function

$$R_X : \mathbb{R} \to \mathbb{R},$$

where $\tau = t_2 - t_1$



The correlation of $X(t_1), X(t_2)$ equals the correlation of $X(t_3), X(t_4)$ since $t_2 - t_1 = t_4 - t_3 = \tau$. In this case, can write $R_X(\tau) = E[X(t)X(t+\tau)]$

Note: the second property of a WSS rp holds if $E[X(t)X(t+\tau)]$ does not depend on t.

Note that if $X(t)$ is WSS, then

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\,\mu_X(t_2)$$

becomes

$$C_{XX}(t_1, t_2) = C_X(\tau) = R_X(\tau) - \mu_X^2$$

Defn. A rp $X(t)$ is a <u>white-noise</u> process if it is WSS and

- $\mu_X = 0$, and
- $R_X(\tau) = N\delta(\tau)$

dirac-delta function

for some $N \in \mathbb{R}$, $N > 0$, where N is called the "noise power"

Comments:

- White noise has

$$E[X(t_1) X(t_2)] = 0$$

if $t_1 \neq t_2$, so $X(t_1)$, $X(t_2)$ are orthogonal for white noise, and thus, since $\mu_X(t) = 0$, $X(t_1)$, $X(t_2)$ are uncorrelated

- The "frequency content" of a WSS rp is defined as the Fourier Transform of $R_X(\tau)$. With this definition, white noise has a constant power over all frequencies, since $\delta(\tau) \overset{\mathcal{F}}{\Longleftrightarrow} 1$ for all frequencies $\omega \in \mathbb{R}$

Linear Time-Invariant Systems with Random Input Signals

From ECE 301: An LTI system can be characterized by its impulse response

$$\delta(t) \longrightarrow \boxed{\text{LTI system}} \longrightarrow h(t)$$

Then

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

What if the input is a rp?

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

This means

$$X(t, w) \longrightarrow \boxed{h(t)} \longrightarrow Y(t, \omega)$$

So for each $w \in S$,

$$Y(t, \omega) = h(t) * X(t, \omega)$$

In practice, often want to know the mean function $\mu_Y(t)$ and the autocorrelation function $R_{YY}(t_1, t_2)$, since these two functions are commonly used to characterize rps:

$$\mu_Y(t) = E\left[Y(t)\right] = E\left[\int_{-\infty}^{\infty} X(t-\alpha)\, h(\alpha)\, d\alpha\right]$$
$$= \int_{-\infty}^{\infty} \underbrace{E\left[X(t-\alpha)\right]}_{\mu_X(t-\alpha)} h(\alpha)\, d\alpha$$

So
$$\boxed{\mu_Y(t) = \mu_X(t) * h(t),\ \forall t}$$

Also,

$$R_{YY}(t_1, t_2) = E\left[Y(t_1)\, Y(t_2)\right]$$
$$= E\left[\int_{-\infty}^{\infty} X(t_1-\alpha)\, h(\alpha)\, d\alpha \int_{-\infty}^{\infty} X(t_2-\beta)\, h(\beta)\, d\beta\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[X(t_1-\alpha)\, X(t_2-\beta)\right] h(\alpha)\, h(\beta)\, d\alpha d_\beta$$

So
$$\boxed{R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(t_1-\alpha, t_2-\beta)\, h(\alpha)\, h(\beta)\, d\alpha d\beta}$$

You should understand how this was derived.

If we assume that $X(t)$ is WSS with $\mu_X(t) = \mu_X, R_{XX}(t_1, t_2) = R_X(\tau)$, with $\tau = t_2 - t_1$

Then $\mu_Y(t) = \int_{-\infty}^{\infty} \mu_X h(\alpha)\, d\alpha$

$$= \mu_X \int_{-\infty}^{\infty} h(\alpha)\, d\alpha = \mu_Y,$$

does not depend on t

Also,

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(t_2 - t_1 - \beta + \alpha)\, h(\alpha)\, h(\beta)\, d\alpha d\beta$$

$$= \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} R_X(t_2 - t_1 + \alpha - \beta)\, h(\beta)\, d\beta\right]}_{(R_X * h)(\tau + \alpha)} h(\alpha)\, d\alpha$$

So

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} (R_X * h)(\tau + \alpha)\, h(\alpha)\, d\alpha$$

To write this more compactly, let $\lambda = -\alpha.$ Then

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(-\lambda)\, (h * R_X)(\tau - \lambda)\, dx$$

Now let $\tilde{h}(t) = h(-t)$. Then

$$\boxed{R_Y(\tau) = \left(\tilde{h} * h * R_X\right)(\tau)}$$

Have shown that if $X(t)$ is WSS, $Y(t)$ has constant mean, and autocorrelation function that depends only on $t_2 - t = \tau$. As long as $\int_{-\infty}^{\infty} h(\alpha)\, d\alpha < \infty$,

$Y(t)$ is WSS.

Example. Let $X(t)$ be a white noise process with

$$R_X(\tau) = N_0 \delta(\tau), N_0 > 0$$

Consider the filter

$$h(t) = e^{-t} u(t)$$

If $X(t)$ is filtered by

$h(t)$, then

$$\mu_Y(t) = \mu_X \int_{-\infty}^{\infty} h(t)\, dt = 0$$

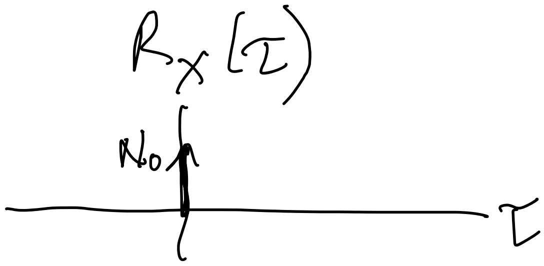
since white noise is zero-mean

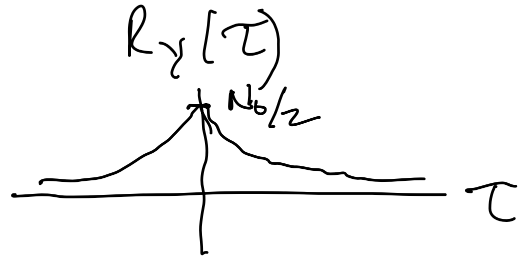
Also,

$$R_y(\tau) = e^{\tau} u(-\tau) * e^{-\tau} u(\tau) * N_0 \delta(\tau),$$

so

$$R_Y(\tau) = \frac{N_0}{2} e^{-|\tau|}; \tau \in \mathbb{R}$$





Note that

- $X(t_1), X(t_2)$ are uncorrelated if $t_1 \neq t_2$, but $Y(t_1), Y(t_2)$ have correlation $\frac{N_0}{2} e^{-|t_2 - t_1|}$

- Filtering a rp can be used to create "colored noise" from white noise. This is often done in music. Also, there are products that deliver white noise and various types of colored noise, created using different filters $h(t)$. Some people report that these products help soothe them and help them sleep, but I do not know if there is any scientific evidence for this.