

Chapter 1

Never Trust an Arctangent

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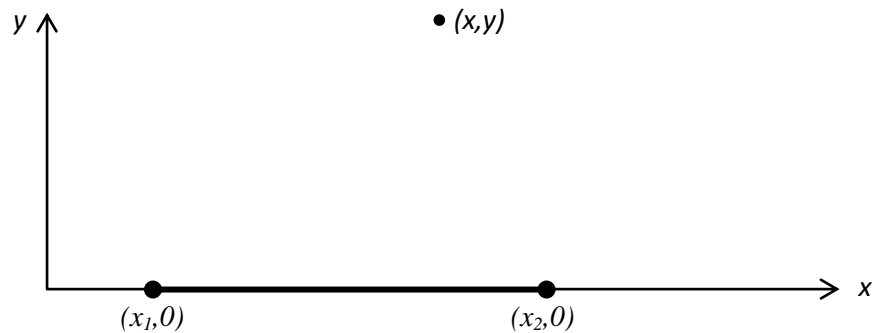
There are a number of reasons to “distrust” an arctangent. That is, there are a number of reasons to be careful when using arctangents in your work. By all means use arctangents where they belong, I’m not recommending anything as an alternative. However, in those places where you do use arctangents, the following reminders or warnings are offered to aid you in securing the correct answer from among one or more seemingly correct wrong answers. Besides, I desired a catchy title for the blog.

I have become reluctantly acquainted with a number of ways of creating errors with arctangents. In this initial blog entry we will examine one of them. For the others, check back in future months. However, I have a variety of topics outlined for the first year so you will need to enjoy a few non-arctangent examples before the next arctangent chapter.

The example in this chapter grows out of basic aerodynamics, yet you need not “know aerodynamics” to appreciate the lesson. As a matter of fact, this sentence here is the last mention of aerodynamics in the chapter. As the formula of interest grew from solutions to Laplace’s equation, applications may be quite diverse.

First off, you may ask, “Distrust arctangents? How can this be? Arctangents are as old as the hills, I learned about them in high school.” Well, the dangers in arctangents are also as old as the hills and they were probably mentioned to you in high school too, but we all forget some details as the years go by and we become rather pleased with the “more advanced” work that we’ve become able to accomplish. Yet one must be skeptical of one’s own use of arctangents. Here are the details: a certain analysis effort brings us to know that the value of a scalar field f at some point (x, y) as sketched below is given, for some real positive constant C , by

$$f(x, y) = C \left[\arctan\left(\frac{y}{x-x_2}\right) - \arctan\left(\frac{y}{x-x_1}\right) \right] \quad (1)$$



Where $x_1 \leq x \leq x_2$ are the two ends of a region of interest, like the leading edge and trailing edge of a wing.

Equation (1) is no problem. Where the current puzzle began is in finding the limit of $f(x, y)$ for $x_1 \leq x \leq x_2$ as $y \rightarrow 0$ first from the positive side and also then the limit as $y \rightarrow 0$ from the negative side. Call these limits:

$$f(x, 0^+) = \lim_{\substack{y \rightarrow 0 \\ y > 0 \\ x_1 < x < x_2}} (f(x, y)) \quad (2)$$

$$f(x, 0^-) = \lim_{\substack{y \rightarrow 0 \\ y < 0 \\ x_1 < x < x_2}} (f(x, y)). \quad (3)$$

You may find it simple to compute these limits by noticing that substituting $y = 0$ into each arctangent of Equation (1) produces a zero argument for each arctangent. As you know that $\arctan(0) = 0$ you could rapidly arrive at

$$f(x, 0^+) = 0 \text{ and} \quad (4)$$

$$f(x, 0^-) = 0. \quad (5)$$

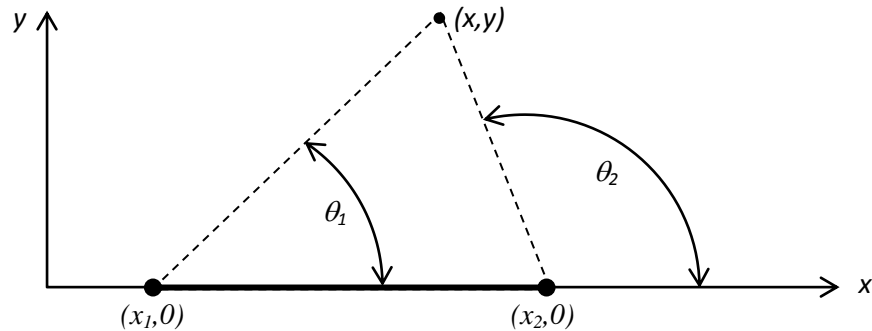
A symbolic processor or a pocket calculator may aid your memory of trig functions to cause you to reach these erroneous results. For example, *Mupad* gives this result:

```
limit(arctan(y/(x-x1)), y=0, Right)
0
limit(arctan(y/(x-x1)), y=0, Left)
0
```

But why are Equations (4) and (5) wrong? Because $\arctan(0) = 0$ only some of the time.

Other times, $\arctan(0) = \pi$. How can this be? Recall that $\sin 0 = 0$ and $\sin \pi = 0$ while $\cos 0 = 1$ and $\cos \pi = -1$, and thus, $\tan 0 = \tan \pi = 0$. It's up to you to "keep track" of the different cases in your analysis, such as by drawing out your geometry in detail. For example, in the sketch below,

visualize how the angles θ_1 and θ_2 each tend towards limiting values when $y \rightarrow 0$ from the positive side for $x_1 \leq x \leq x_2$.



You will see that $\theta_1 \rightarrow 0$ and $\theta_2 \rightarrow \pi$. Thus, Equation (2) becomes

$$f(x, 0^+) = C[\pi - 0] = C\pi \quad (6)$$

To find the limit for $y < 0$, draw a triangle that connects the points $(x_1, 0)$, $(x_2, 0)$, and (x, y) for the case where $y < 0$ and $x_1 \leq x \leq x_2$. From this triangle, as $y \rightarrow 0$ for $x_1 \leq x \leq x_2$ you can see that Equation (3) becomes

$$f(x, 0^-) = C[-\pi - 0] = -C\pi \quad (7)$$

Thus, carelessness with arctangents when the argument tends toward zero can lead to one of several possible wrong answers.

To solve the problem properly in *Mupad*, you make use of two features. The first feature is the “assume” command. The second is the two-argument version of arctangent function (and other packages doubtless have analogous capabilities.) Specifically:

```
assume(x>x1 and x<x2)
limit(C*(arctan(y, x-x2)-arctan(y, x-x1)), y=0, Right)
π C
limit(C*(arctan(y, x-x2)-arctan(y, x-x1)), y=0, Left)
-π C
```

The arctangent function called with two arguments (that is, $(y, x-x1)$, and note the comma, instead of $(y/(x-x1))$ with the solidus or “division symbol” or “forward slash”) provides you a four-quadrant answer, which is needed in this and other geometries that you may be faced with. But to solve these problem correctly in *Mupad* or other packages, you the user must think about the geometry of the problem and choose to pay attention to the details. It’s your effort that will bring you to the correct answer regardless of the mode: paper and pencil, symbolic manipulation, or numerical.

Some lessons one might generalize from the above discussion are:

1. Arctangent is a tool but you're a thinker. *Mupad, Mathematica*, etc., are tools but you have a brain and can see the "big picture." Never surrender your innate advantage over the mindless tools. Think.

2. Sketch the geometry that you are working with. Ponder it. Consider changes to it that the constraints in your problem may create.

3. All four quadrants are relevant to working with arctangents.

4. Never trust an arctangent. Therefore, verify the results that you derive or compute. In the end, any errors will be your fault, not the fault of the arctangent.

Want to test your understanding? Try this similar task:

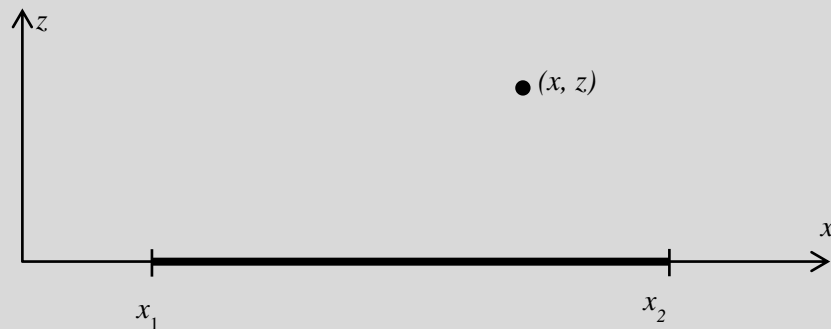
Never Trust an Arctangent study problem #1

Given a positive real constant C , a certain dependent variable field is solved by integration to be

$$w(x, z) = C \left[\arctan \frac{x - x_o}{z} \right]_{x_1}^{x_2}. \text{ Here the limits of integration, } x_1 \text{ and } x_2, \text{ are to be substituted for}$$

the x_o symbol in the argument of the arctangent.

- Find the limit of $w(x, z)$ for $x_1 < x < x_2$ and $z \rightarrow 0^+$, that is, from above.
- Find the limit of $w(x, z)$ for $x_1 < x < x_2$ and $z \rightarrow 0^-$, that is, from below.



Some other arctangent concerns intended to be explored in later chapters:

- The analytical relation: $\arctan(-x) = -\arctan(x)$ is also dangerous as it can destroy quadrant information if poorly applied.
- Also be wary of the typical **atan**(\mathbf{y}/\mathbf{x}) numerical function in computations. The use of **atan2**(\mathbf{y}, \mathbf{x}), which is the four-quadrant arctangent and is available in most every package or programming language, is generally safer.

- With either a numeric arctangent function, with symbolic processors, and in pencil-pushing analyses, there is always a branch cut somewhere in the (x, y) plane for the arctangent.