From Stopping sets to Trapping sets

The Exhaustive Search Algorithm & The Suppressing Effect

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Good exhaustive trapping set search algorithm for arbitrary codes.

The suppressing effect for cyclically lifted code ensembles.
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New results on the hardness of the problem

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Content

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  - Existing work on exhaustive search for stopping sets

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- The exhaustive search for trapping sets based on exhaustive search for stopping sets.

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Quantifying the suppressing effect.
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The **suppressing effect** for cyclically lifted **code ensembles**.

- Definition: $\text{Prob}(\text{the bad structure remains after lifting})$
- **Quantifying** the suppressing effect.
- A design criteria for **base code optimization**.
Stoppers Sets

Definition: a set of variable nodes \( \Rightarrow \) the induced graph contains no check node of degree 1.

\[ \begin{array}{c}
 i & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 j & 4 & 2 & 3 \\
\end{array} \]
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- Error floor optimization. BECs vs. non-erasure channels.

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Good but inexhaustive search algorithms: error floors of LDPC codes [Richardson 03], projection algebra [Yedidia et al. 01], the approximate minimum distance of LDPC codes [Hu et al. 04], [Hirotomo et al. 05], [Richter 06]
An NP-Hard Problem

=== The $SD(H, t)$ problem ===

**INPUT:** A code represented by its parity-check matrix $H$ and an integer $t$.

**OUTPUT:** Output 1 if the minimal stopping distance of $H$ is $\leq t$. Otherwise, output 0.

The hardness results:

- [Krishnan et al. 06]: For arbitrary $H$, $SD(H, t)$ is NP-complete.
  
  Proof: By reducing a VERTEX-COVER problem to $SD(H, t)$.

- A byproduct of [Krishnan et al. 06]: With the sparsity restriction that the number of 1’s in $H$ is limited to $O(n)$ rather than $O(n^2)$, then $SD(H, t)$ is still NP-complete.
Trapping Sets: Definitions

Operational definition: “the set of bits that are not eventually correct” [Richardson 03].
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  - A \((a, 0)\) near codeword \(\not\Rightarrow\) a stopping set
Trapping Sets: Definitions

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- We propose a new graph-theoretic definition:

  **Definition 1 (\(k\)-out Trapping Sets)** A subset of \(\{v_1, \ldots, v_n\}\) such that in the induced subgraph, there are exactly \(k\) check nodes of degree one.
**\( k \)-Out Trapping Sets vs. Near-Codeword**

**Definition 1 (\( k \)-out Trapping Sets)** A subset of variables such that in the induced subgraph, there are exactly \( k \) check nodes of degree one.

- \( k \)-out trapping sets \( \leftrightarrow \) stopping sets
  \((a, b)\) near-codewords \( \leftrightarrow \) valid codewords

- 0-out trapping sets \( \leftrightarrow \) stopping sets
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**k-Out Trapping Sets vs. Near-Codeword**

**Definition 1** (*k*-out Trapping Sets)  A subset of variables such that in the induced subgraph, there are exactly *k* check nodes of degree one.

- *k*-out trapping sets $\iff$ stopping sets
  $(a, b)$ near-codewords $\iff$ valid codewords

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**Why this definition?**

- Better analogy to stopping sets.
- An \((a, b)\) near-codeword \( \neq \) "\( k \leq b\)"-out trapping set.
- Our goal: With fixed \( b \), search all min. \( k \leq b\)-out TSs.
**k-OUT TRAPPING SETS VS. NEAR-CODEWORD**

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- An $(a, b)$ near-codeword $\iff$ “$k \leq b$"-out trapping set.
- Our goal: With fixed $b$, search all min. $k \leq b$-out TSs.
- Empirically, all error bits consist of only degree 1 & 2 check nodes. (The elementary trapping set [Landner et al. 05].)
The Hardness of $k$-OTD($H, t$)

=== The $k$-OTD($H, t$) problem ===

**INPUT:** A code represented by its parity-check matrix $H$ and an integer $t$.

**OUTPUT:** Output 1 if the minimal $k$-out trapping distance of $H$ is $\leq t$. Otherwise, output 0.
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- When $k = 0$, then $0$-OTD$(H, t) = SD(H, t)$ is NP-complete.
- Is the hardness the same for any fixed $k > 0$ values?
Our First Result

**Theorem 1**  Consider a fixed $k > 0$. For arbitrary $H$, $k$-OTD($H, t$) is **NP-complete**.

**Theorem 2**  Consider a fixed $k > 0$. With the *sparsity restriction* that the number of 1’s in $H$ is limited to $O(n)$ rather than $O(n^2)$, then $k$-OTD($H, t$) is **still NP-complete**.

Proof: Reduction from SD($H, t$).
SD\((H, t)\) By \(k\)-OTD\((H', t')\)

Step 1: Duplicate \(G\) \((k + 2)\) times

\(k = 2\)
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Run $k$-OTD$(H', t(k + 2))$. 

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- and it must contain the target bit.
**SD(\(H, t\)) By \(k\)-OTD(\(H', t'\))**

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**Is there anything else we can do?**
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*Is there anything else we can do?*

- NP-completeness $\implies$ the **asymptotic complexity**.
- NP-completeness has relatively less predictability for finite $n$.
- For practical codes, we only need $n \approx 500–5000$. 

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NP-hard problem = Impossible?

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Leverage Upon $\text{SD}(H, t)$

=== The $\text{SD}(H, t)$ problem ===

**OUTPUT:** Output an exhaustive list of minimum stopping sets if the minimal stopping distance is $\leq t$. Otherwise, output $\emptyset$.

- In our previous work [ISIT 06], a good exhaustive search $\text{SD}(H, t)$ is provided.
- Capable of exhausting $t = 11–13$ for codes of $n \approx 500$. 
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- On this Friday 4:45pm [Rosnes & Ytrehus, ISIT07], a more efficient exhaustive search $SD(H, t)$ will be introduced.
  - Capable of exhausting $t = 18–26$ for codes of $n = 150–5000$. 
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- Good SD($H, t$) $\Rightarrow$ good $k$-OTD($H, t$)
$k$-OTD$(H, t')$ By SD$(H, t)$

$k = 2$
1. Select \( k \) edges.
$k$-OTD$(H, t')$ By SD$(H, t)$

1. Select $k$ edges.
2. Based on the $k$ check nodes, identify the neighbor variables.
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$k$-OTD($H, t'$) By SD($H, t$)

1. Select $k$ edges.

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3. Remove the check nodes and neighbor variables.

4. Run SD($H, t$) to find the minimal stopping sets containing the interested variables.
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2. Based on the $k$ check nodes, identify the neighbor variables.

3. Remove the check nodes and neighbor variables.

4. Run $\text{SD}(H, t)$ to find the minimal stopping sets containing the interested variables.

5. Select another $k$ edges and repeat the procedure.
Empirical Study of $k$-OTD($H, t$)

- Complexity grows $O(n^k)$. 
Empirical Study of $k$-OTD$(H, t)$

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- Complexity grows $O(n^k)$. A harder problem than SD$(H, t)$.
- For codes of interest, 50% FER from $k \leq 2$ TS [Richardson 03].
- When $n \approx 500$ and rate $\frac{1}{2}$ codes, $t = 10–12$ for 1-OTS$(H, t)$.
  $t = 9–11$ for 2-OTS$(H, t)$, based on our SD$(H, t)$.
Empirical Study of $k$-OTD($H, t$)

- Complexity grows $O(n^k)$. A harder problem than SD($H, t$).
- For codes of interest, 50% FER from $k \leq 2$ TS [Richardson 03].
- When $n \approx 500$ and rate $\frac{1}{2}$ codes, $t = 10–12$ for 1-OTS($H, t$).
  \hspace{1cm} $t = 9–11$ for 2-OTS($H, t$), based on our SD($H, t$).
- **Tanner (155,64,20) code 04**: Minimal 1-out TD $\geq 12$, and \hspace{1cm} minimal 2-out TD $= 8$ w. multiplicity 465.
  All from the following by automorphisms [Tanner et al. 04].

\begin{align*}
7, 17, 19, 33, 66, 76, 128, 140 \\
7, 31, 33, 37, 44, 65, 100, 120 \\
1, 19, 63, 66, 105, 118, 121, 140 \\
44, 61, 65, 73, 87, 98, 137, 146 \\
31, 32, 37, 94, 100, 142, 147, 148.
\end{align*}
Empirical Study of $k$-OTD($H, t$)

- **Ramanujan-Margulis (2184,1092) Code** w. $q = 13$, $p = 5$ [Rosenthal et al. 00];
- **Inexhaustive results — upper bounds**: analytical search [Mackay et al. 03], error-impulse search [Hu et al. 04]

  Minimum Hamming distance $\leq 14$

- **Exhaustive results by SD($H, t$) — lower bounds**:

  Minimum Hamming distance $\geq$ minimum SD $\geq 14$
  multiplicity 1092

Min. 1-out TD $\geq 13$ and min. 2-out TD $\geq 10$. 
\[ \lambda(x) = 0.31961x + 0.27603x^2 + 0.01453x^5 + 0.38983x^6, \quad \rho(x) = 0.50847x^5 + 0.49153x^6 \]

AWGN, \((\lambda(x), \rho(x))\), \(n = 512\), 0-out/1-out trapping sets.

“Rand” \((2, 1), (2, 8)\); “SS Opt” \((13, 40), (5, 4)\); “SS+TS Opt” \((11, 12), (10, 24)\).

Sum-product decoder, 80 iterations, 100 frame errors.
Insufficiency of TSs

The relationship to error floors.

- $n = 504$ Girth-optimized Irregular PEG code [Hu et al. 05], 1-out TSs of size 7:
  
  52, 53, 122, 136, 178, 229, 348
  5, 42, 100, 131, 187, 199, 374

- $n = 504$ TS-optimized irregular code w. the same deg. distr., 0/1-out TSs: $(10, 7)/(8, 40)$. 
Insufficiency of TSs

The relationship to error floors.

$n = 504$ Girth-optimized
1-out TSs of size 7:
52, 53, 122, 136, 178, 229, 348
5, 42, 100, 131, 187, 199, 374

$n = 504$ TS-optimized code
0/1-out TSs: $(10, 7)/(5, 42)$

Frame / Bit Error Rate (FER/BER)

Signal to Noise Ratio: $E_s/N_0 = 20\log(1/\sigma)$
The Cyclically Lifted Ensemble

[Gross 74], [Richardson & Urbanke] and many more.

(a) The base code  (b) The lifted code with an all-zero lifting sequence

(c) The lifted code with a cyclic lifting sequence.
The Cyclically Lifted Ensemble

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Based Code Optimization $\Rightarrow$ lower ensemble error floor.

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(c) The lifted code with a cyclic lifting sequence.

Base Code — of size $n$ ($n = 16$)

Lifted Code — of lifting factor $K$ ($K = 4$)
Theorem 3  If ⬤ forms a $k_L$-out trapping set for one lifted code, then ⬤ forms a $k_B$-out trapping set for the base code where $k_L \geq k_B$. 

Base Code — of size $n$ ($n = 16$)

Lifted Code — of lifting factor $K$ ($K = 4$)
Different Orders of Survivals

Definition 2

First order survivals

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\[
\begin{array}{cccccccc}
\cdot \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\end{array}
\]

Lifted Code — of lifting factor $K$ ($K = 4$)

\[
\begin{array}{cccccccc}
\cdot \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\end{array}
\]

Definition 3

High order survivals

Base Code — of size $n$ ($n = 16$)

\[
\begin{array}{cccccccc}
\cdot \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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Lifted Code — of lifting factor $K$ ($K = 4$)

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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]
## Different Orders of Survivals

**Definition 2**

*First order survivals*

- **Base Code** — of size $n$ ($n = 16$)
- **Lifted Code** — of lifting factor $K$ ($K = 4$)

**Definition 3**

*High order survivals*

- **Base Code** — of size $n$ ($n = 16$)
- **Lifted Code** — of lifting factor $K$ ($K = 4$)

Empirically, *almost all small trapping sets are of first order.*

[Wang 06, Ländner 05]
Theorem 4 (\(k_L = k_B = 0\), a preliminary result)  For a fixed base code with a min. stopping set \(s_B\),

\[
E\{|\text{first order survivals}|\} \propto K^{- \left(0.5\#E - \#V + 0.5\#C_{\text{odd}}_{\geq 3}\right)}
\]

\[
\text{FER}_{\text{BEC, ensemble}} = \text{const} \cdot K^{- \left(0.5\#E - \#V + 0.5\#C_{\text{odd}}_{\geq 3}\right)}.
\]

where \(\text{const} = f(\text{the min. stp. dist., multi.})\).
First order survival

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where \(\text{const} = f(\text{the min. stp. dist., multi.})\).

Theorem 5 (\(k_L = k_B > 0\)) For a base-code \(k\)-out trapping set \(t_B\),
\[
E\{|\text{first order survivals}|\} \propto K^{0.5k_B} K^{-\left(0.5\#E - \#V + 0.5\#C_{\text{odd}}\geq 3\right)}.
\]
First order survival

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E\{|\text{first order survivals}\}| \propto K^{-\left(0.5\#E - \#V + 0.5\#C_{\text{odd,} \geq 3}\right)}
\]

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F_{\text{FER BEC,ensemble}} = \text{const} \cdot K^{-\left(0.5\#E - \#V + 0.5\#C_{\text{odd,} \geq 3}\right)}.
\]

where \(\text{const} = f(\text{the min. stp. dist., multi.})\).

Theorem 5 \((k_L = k_B > 0)\) For a base-code k-out trapping set \(t_B\),

\[
E\{|\text{first order survivals}\}| \propto K^{0.5k_B} K^{-\left(0.5\#E - \#V + 0.5\#C_{\text{odd,} \geq 3}\right)}.
\]

Theorem 6 \((k_L = k_B + 1)\) For a base-code k-out trapping set \(t_B\),

\[
E\{|\text{first order survivals}\}| \propto K^{0.5k_B} K^{-\left(0.5\#E - \#V + 0.5\#C_{\text{odd,} \geq 3}\right)} (K\#C_{\text{odd,} \geq 3} + \#C_{\text{even,} \geq 4}).
\]
First order survival

Theorem 4 ($k_L = k_B = 0$, a preliminary result) For a fixed base code with a min. stopping set $s_B$,

$$E\{|\text{first order survivals}|\} \propto K^{-\left(0.5E - #V + 0.5C_{odd, \geq 3}\right)}$$

$$FER_{BEC, ensemble} = \text{const} \cdot K^{-\left(0.5E - #V + 0.5C_{odd, \geq 3}\right)}.$$  

where const $= f(\text{the min. stp. dist., multi.})$.

Theorem 5 ($k_L = k_B > 0$) For a base-code $k$-out trapping set $t_B$,

$$E\{|\text{first order survivals}|\} \propto K^{0.5k_B} K^{-\left(0.5E - #V + 0.5C_{odd, \geq 3}\right)}.$$  

Theorem 6 ($k_L = k_B + 1$) For a base-code $k$-out trapping set $t_B$,

$$E\{|\text{first order survivals}|\} \propto K^{0.5k_B} K^{-\left(0.5E - #V + 0.5C_{odd, \geq 3}\right)} (K#C_{odd, \geq 3} + #C_{even, \geq 4}).$$

Base code optimization: $0.5E - #V + 0.5C_{odd, \geq 3}$
First order survival

Theorem 4
For a fixed base code with a min. stopping set $B$, $E\{|f\} \propto K - (0.5\#E - \#V + 0.5\#C_{odd, \geq 3})$.

Theorem 5
For a $k$-out trapping set $t_B$, $E\{|f\} \propto K^{0.5}K - (0.5\#E - \#V + 0.5\#C_{odd, \geq 3})$.

Theorem 6
For a $k$-out trapping set $t_B$, $E\{|f\} \propto K^{0.5}K - (0.5\#E - \#V + 0.5\#C_{odd, \geq 3})$.

Base code optimization: $0.5\#E - \#V + 0.5\#C_{odd, \geq 3}$.

$E_S/N_0 = 20\log(1/\sigma)$

Frame / Bit Error Rate (FER/BER)

$E_B = 128$, $K = 4$. $0/1$-out TSs: (11,12)/(10,24)

$\#C_{even, \geq 4}$.
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Implement $k$-OTD$(H, t)$ by SD$(H, t)$.
Conclusion

- Define the $k$-out trapping set graph-theoretically.
- Deciding the minimal $k$-out trapping distance is \textbf{NP-hard}.
- But still doable for practical code lengths $n \approx 500$.
- Implement $k$-OTD$(H, t)$ by $SD(H, t)$.
- Insufficiency of the trapping set (near-codeword).
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