Density Evolution
for Asymmetric Memoryless Channels

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Belief Propagation:

(i) $m_0, (ii) \mathbf{v} \left( m_0, m_1, \ldots, m_{d-1} \right),$ (iii) $\mathbf{c} \left( m_1, \ldots, m_{d-1} \right)$. Note: the belief propagation is an inference algorithm, which is independent of channel models.

Figure 1: $\mathcal{C}_6 \left( \mathcal{C}_6 \right)$ and $p \left( \mathcal{C}_6 \right)$ and $p \left( \mathcal{C}_6 \right) = 3$. $d = 2, \ d = 3, p = 3$. "Graph Ensemble x Belief Propagation"
\[ (1 - a \rho) \otimes (1 - l) \mathcal{O} = (1 - l) \mathcal{O} \]

\[ (1 - a \rho) \otimes (1 - l) \mathcal{O} \otimes (0) d = (1) d \]
\[(I-\alpha p) \otimes ((x)(I-1)d) \mathcal{E} \mathcal{D} = (x)(I-1)\mathcal{D} \]

\[(I-\alpha p) \otimes ((x)(I-1)\mathcal{D}) \otimes (x)(0)d = (x)(I)d\]

Codeword Dependence
Difficulty of Codeword Averaging

Averaging of the trimmed tree code is not equivalent to averaging the original code.

\[
\{000,011,101,110\} = 9x^3x1x
\]

\[
: 6x^3x1x = (1^1)X
\]
\[ \left( \left( (1)_{(t)}d \right) p \int_0^\infty + \left( (0)_{(t)}d \right) p \int_0^0 \right) \frac{2}{I} = (t)_{(l)}d \]

and

\[ \{ x = 0 | x: X \subseteq x \} \]

\[ \langle (\hat{x})_{(t)}d \rangle = \langle (x)_{(l)}d \rangle =: (x)_{(l)}d \]

Suppose the supporting tree \( N_{2l} \) is of full rank. We have

\[
\frac{|\hat{x}|}{I} = \frac{|X|}{|\{ \hat{x} = \text{tree} | x: X \subseteq x \}|}
\]

if for any codeword \( x \) of the tree code \( X \), the supporting tree \( N_{2l} \) is of full rank. We have

**Definition 1 (Full Rank Condition)**

Full Rank Condition
\[
\left( \left( \begin{array}{c}
\binom{a}{x}
\end{array} \right)_{(1-1)d} \right) \bigotimes_{1-\alpha p} \left( \begin{array}{c}
\binom{a}{x} \\
\binom{a}{x}
\end{array} \right)_{(1-1)\hat{\mathcal{D}}}
\right)
\bigotimes_{1-\alpha p} \left( \begin{array}{c}
\binom{a}{x}
\end{array} \right)_{(0)d} = \left( \begin{array}{c}
\binom{a}{x}
\end{array} \right)_{(1)d}
\]

\[
\begin{align*}
01, 10 &= \binom{a}{x} \\
11, 00 &= \binom{a}{x}
\end{align*}
\]

\[
\begin{align*}
01, 10 = \binom{a}{x} \\
11, 00 = \binom{a}{x}
\end{align*}
\]
\[ \begin{align*}
\left( \left( \frac{\mathcal{I}}{(1-(I-1))d - (0)} \right) \cdot (I-1) \right) x(I-1) + \\
\left( \left( \frac{\mathcal{I}}{(1-(I-1))d + (0)} \right) \cdot (I-1) \right) I = (x)_{(I-1)}d \\
\left( \left( (x)_{(I-1)}d \right) \otimes (x)_{(0)}d \right) = (x)_{(1)}d
\end{align*} \]
Convergence to Full Rank.

Theoretical foundations for the codeword averaging approach:

For any \( \gamma \) in Probability, with fixed \( l \), we have

\[ P \left( N_{\gamma l} \text{ is of full rank} \right) = 1 - O\left( \frac{1}{n^{\gamma - 0.1}} \right). \]

\[ P \]

Theorem I (Convergence to Full Rank in Probability)
Finite Codeword Length Simulations

12A: [Richardson01], optimized BiAWGNC, maxDv=12.
12B: Optimized z-channels, maxDv=12, λ_2 = 0.
12C: Optimized z-channels, maxDv=12.

Symmetrized Rate

(3,6) of 12B
(3,6) of 12A
λ_2 = 0.

Bit error probability

p.9

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Density Evolution

Number of Iterations

bit error probability

$P_e(0)$

$P_e(1)$

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Monotonicity & Symmetry

Monotonicity of

\[ p(l) = (p(l)_{0} + p(l)_{1}) = 2 \]

with respect to the number of iterations and to physically degraded channels.

Symmetry:

\[ \mathcal{I}(p) = \mathcal{I}(\eta) \]

is the Bhattacharyya noise parameter.

\[ \langle (1) d \rangle \]

is the Chernoff version of \( \langle (1) d \rangle \), is the Chernoff version of \( \langle (1) d \rangle \).

\[ \eta = \langle (1) d \rangle \]

is symmetric. I.e. it is as if being obtained from a symmetric channel.

\[ \mathcal{H}_{CBP}(l) \]

is the Chernoff version of \( \mathcal{H}_{P}(l) \).

\[ \mathcal{H}_{CBP}(0) \]

is the Bhattacharyya parameter.

\[ \langle (h|0=x)d \rangle \]

is the distribution of \( m \).

\[
\text{Symmetry:}
\]

\[
\text{Monotonicity of}
\]

\[
\text{Symmetry} \Rightarrow \text{Symmetry}
\]
Proposition:

\[ \langle (l) p \rangle C B P(l) = 0. \]

which immediately implies \( \lim_{l \to \infty} h \) is

\[ \{ 0 < \forall x : \min = 0 \}, \quad \forall x \in \mathbb{R}, \quad 0 = \exists x \ni \left( \left( (l-\forall x) \right) \right) 0. \]

\[ 0 < \exists x \ni \left( \left( (l \forall x) \right) \right) 0. \]

If for certain \( l \), then

\[ \varepsilon = \mu(l_1) \forall x. \]

Theorem 2 (Sufficient Stability Condition)

Suppose \( \exists \varepsilon \forall x : \min > 1 \), and

let \( \varepsilon \) be the smallest strictly positive root of the following equation.

\[ \mu(l_1) \forall x. \]

Let \( \forall x \exists \varepsilon \forall x : (x) \forall x \subseteq \exists (x) \forall x \subseteq \exists \exists \forall x \subseteq \exists (x) \forall x. \]

Proposition:

Sufficient Stability Conditions
Sufficient Stability Conditions
Theorem 3 (Necessary Stability Condition)

\[ r := h_{CBP}(0). \]

If \[ \rho \geq 0 \] and

\[ r > 1, \]

then

\[ \lim_{l \to 1} p_l(\rho) > 0. \]

Note: It is first stated in [Richardson et al. 01] for symmetric channels.
Sketches of the Sufficient Stability Condition

\[ \langle (1) \rho \mathcal{C} \rho \mathcal{B} \mathcal{P} \rangle (\mathcal{I}, \theta) \mathcal{C} \langle (0) \rho \mathcal{B} \mathcal{P} \rangle \geq \langle (1+1) \rho \mathcal{B} \mathcal{P} \rangle \]

Irregular codes:

\[ \cdot \left( \langle (1) \rho \mathcal{C} \rho \mathcal{B} \mathcal{P} \rangle (\mathcal{I} - \theta \rho) \right) \langle (0) \rho \mathcal{B} \mathcal{P} \rangle \geq \langle (1+1) \rho \mathcal{B} \mathcal{P} \rangle \]

Regular codes:

\[ \mathcal{C} \mathcal{B} \mathcal{P} = \mathcal{Z} \rho \times \mathcal{I} \rho \left( \frac{z/w_1 \mathcal{C} + z/w_2 \mathcal{C}}{z/w_1 + z/w_2} \right) \int \geq \mathcal{C} \mathcal{B} \mathcal{P} \]

\[ \mathcal{Z} \rho \times \mathcal{I} \rho \left( \frac{z/w_1 \mathcal{C} + z/w_2 \mathcal{C}}{z/w_1 + z/w_2} \right) \int = \partial \mathcal{P} \frac{z/w_1 \mathcal{C} + z/w_2 \mathcal{C}}{z/w_1 + z/w_2} \int = : \partial \mathcal{C} \mathcal{B} \mathcal{P} \]
Sketches of the Sufficient Stability Condition

Stronger version:

\[ h_{CBP}(l+1)i \cdot h_{CBP}(0)i \geq h_{CBP}(l)i \cdot \frac{1}{4} \cdot \frac{1}{4} \]

Comparison between BEC:

\[ \langle (l)p_{CB} \rangle \cdot (\langle (l+1)p_{CB} \rangle - 1) \cdot \langle (0)p_{CB} \rangle = \langle (l+1)p_{CB} \rangle \]

Applications of the sufficient stability condition:

- Stopping time for the density evolution
- Finite dimensional iterative formula
- Analytical upper threshold computation: BEC, Gallager's decoding algorithm
- Algorithm A [Bazzi and Richardson]
By the convergence rate results and the union bound, we prove that it

Convergence Rate and Block Error Probability

\[ \lim_{n \to \infty} P_e(n) = 0. \]
Thresholds for Different Channels

Our best codes for z-channels: max $d = 20$ and max $c = 10$.

• Hitting $\frac{d}{c}$ within 100 iterations: 0.2741.

• Hitting $\frac{d}{c}$ within 500 iterations: 0.2796.

• Symmetric Information Rate bound: 0.2932.

• Capacity bound: 0.3035.
Conclusions

A new iterative formula of density evolution for asymmetric memoryless channels with the same computational complexity.

Good codes for z-channels have been found. Rate 1/2 code: 0.2796 vs. symmetric mutual info. Rate: 0.2932.

LDPC codes plus belief propagation algorithms work well even in asymmetric memoryless channels as expected and theoretically shown.

The stability region/stopping criterion and asymptotic convergence rates have been found and are fully specified by $\lambda$, $d$, and the Bhat-tacharyya noise parameter.

The new density evolution algorithm works well even in asymmetric memoryless channels with the same computational complexity.
Conclusions

By focusing on its Chernoff domain, a one-dimensional universal upper bound is provided and with similar form to BEC thresholds. The convergence behavior of the block error probability is discussed.

Applications beside z-channels and other asymmetric channels:

- The analysis of artificial channels. Example: from q-ary to binary.
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