* The connection between DTFT & the Z transform.

Analysis formula

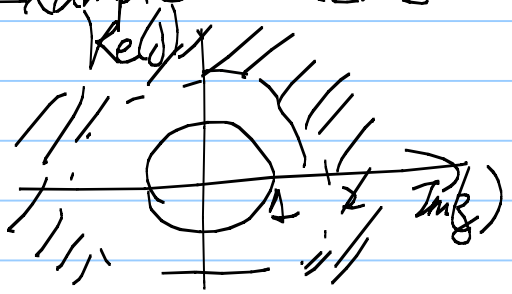
$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\left(e^{jw}\right)^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n](z)^{-n}$$

* Implication 1: If we know the expression of $X(z)$ & its ROC. And If the ROC also contains $e^{jw} = z$, the unit circle, then we can evaluate the DTFT by pluging in $z = e^{jw}$

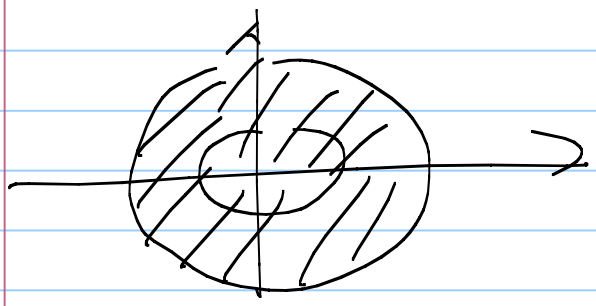
* Also, the DTFT exists if the ROC of $X(z)$ includes the unit circle.

Example: $x[n] = 2^n u[n]$



the ROC does not contain the unit circle. $\Rightarrow$ No DTFT

$x[n] = -2^n u[-n-1]$



$\Rightarrow$ ROC contains the unit circle & $X(z) = \frac{1}{1-2z^{-1}}$

$$\Rightarrow \text{DTFT } X(e^{j\omega}) = \frac{1}{1-2e^{-j\omega}} \quad *$$

* It is critical to check whether the ROC contains the unit circle before using the Z-transform to derive the DTFT

* Inverse $\mathcal{Z}$ transform: Using the inverse DTFT to find out the inverse $\mathcal{Z}$-transform

$$X(\mathcal{Z}) = \sum_{n=-\infty}^{\infty} x[n](\mathcal{Z})^{-n}$$

If $\mathcal{Z} = re^{j\omega}$ is in the ROC.

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\left(r^{-n}e^{-j\omega n}\right)$$

$$= \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right)e^{-j\omega n}.$$

$$\Rightarrow \mathcal{F}^{-1}\left(X(re^{j\omega})\right) = x[n]r^{-n}$$

$$\Rightarrow x[n] = r^{n}\mathcal{F}^{-1}\left(X(re^{j\omega})\right)$$

Detailed steps:

Step 1: Find an arbitrary r value such that $|\mathcal{Z}| = r$ circle is in the ROC. Fix that r.

Step 2: Use $X(\mathcal{Z})$ to construct

$$Y(e^{j\omega}) = X(re^{j\omega})$$

Step 3: Find the Inverse DTFT of $Y(e^{j\omega})$ $\rightarrow y[n]$

Step 4: $x[n] = \gamma^n y[n]$.

Example: $X(z) = \frac{1}{1-2z^{-1}}$, ROC: $|z|>2$. Find $x[n]$.

Ans: Solution 1: Table look up

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Solution #2:

Step 1: choose $\gamma = 3$

Step 2: $Y(e^{j\omega}) = \frac{1}{1-2(3^{-1}e^{-j\omega})}$

$= \frac{1}{1-\frac{2}{3}e^{-j\omega}}$.

Step 3: Inverse DTFT of $Y(e^{j\omega})$

$\Rightarrow y[n] = \left(\frac{2}{3}\right)^n u[n]$

Step 4: $x[n] = 3^n y[n] = 2^n u[n]$.

Example: $X(z) = \frac{1}{1-2z^{-1}}$, ROC: $|z|<2$. Find $x[n]$.

Ans: Step 1: Choose $\gamma = 1$

Step 2: $Y(e^{j\omega}) = \frac{1}{1-2e^{-j\omega}}$

Step 3: Find $y[n]$.

DTFT Table look-up?

$y[n] = 2^n u[n]$ Not working

since the table

$a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\omega}}$

requires $|a|<1$.

This technique can also be used to solve Spring 08 MT3 Q3.

$$Y(e^{j\omega}) = \frac{\frac{1}{-2e^{-j\omega}}}{1-\frac{1}{2}e^{j\omega}} = e^{j\omega} \cdot \underbrace{\frac{-\frac{1}{2}}{1-\frac{1}{2}e^{+j\omega}}}_{Z(e^{j\omega})}$$

$\Rightarrow Z(e^{j\omega}) \rightarrow$

$z[n] = -\frac{1}{2}\left(\frac{1}{2}\right)^{-n} u[-n]$

$y[n] = z[n+1] = -\frac{1}{2}\left(\frac{1}{2}\right)^{-n-1} u[-n-1]$

$= -2^n u[-n-1]$