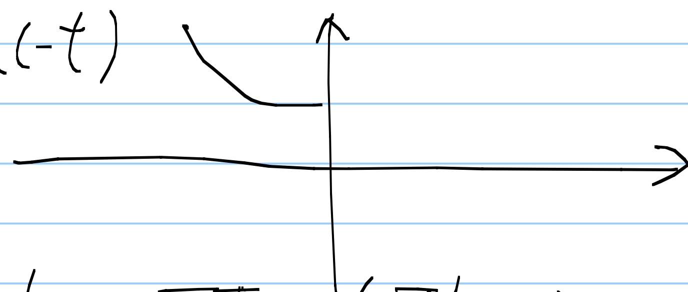
Section 10: The Z-transform

FT is a powerful analytical tool, but it has its limitations.

For example:

$e^{t}u(t)$

& $e^{-t}u(-t)$



do not have FT. (The integration

$$\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{t(1-j\omega)}\,dt$$

blows up since $\int_{0}^{\infty} e^{rt}\,dt$ exists

only if $\underline{\underline{Re(r) < 0}}$.

How do we do freq analysis for signals that blow up.

We need more general, more powerful tools:

CT signals: the Laplace Transform

DT signals: the Z transform

* Unfortunately, more powerful usually means less straightforward than the FT. We need to take into account new concepts like "Region of Convergence (ROC)"

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* Digital Signal Processing is important. Let us focus on the Z-transform.
* Here is our approach.

Consider $x[n] = a^n u[n]$, ($|a|>1$) Suppose we kill the <u>blowing up</u> aspect of $x[n]$ by "exponential time weighting". Instead of $a^n u[n]$, consider $\underline{\underline{\frac{1}{r^n}}} a^n u[n]$ where

$\left|\frac{1}{r}\right| \cdot |a| < 1$. Then the new weighting signal does have a DTFT

$$\sum_{n=-\infty}^{\infty} \left(\frac{a}{r}\right)^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n e^{-j\omega n} = \frac{1}{1 - \frac{a}{r} e^{-j\omega}}$$

Then as long as we remember that we are dealing with an $(\frac{1}{r})^n$ weighting, we should be able to carry through the Fourier analysis.

Let us examine more closely the FT with exponential time weighting $\underline{\underline{(\frac{1}{r})^n}}$

$\Rightarrow$ The DTFT is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, r^{-n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] (r e^{+j\omega})^{-n}$$

Denoting $z = r e^{j\omega}$, we thus have

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$\hookrightarrow$ $X(z)$ is termed the Z-transform analysis formula

* Z transform

subject : discrete-time signals $x[n]$ that may "blow-up".

Formula:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(1+j) = X(\sqrt{2} e^{j\frac{\pi}{4}})$$

exponentially weighted by $(\frac{1}{\sqrt{2}})$

$$= \sum_{n=-\infty}^{\infty} \left(x[n] \cdot \frac{1}{\sqrt{2}^{n}}\right) e^{-j\frac{\pi}{4}n}$$

Compare to DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

☆ It is as if we attenuate $x[n]$ by $\frac{1}{\sqrt{2}^{n}}$ & then perform DTFT & evaluate it at $\omega = \frac{\pi}{4}$