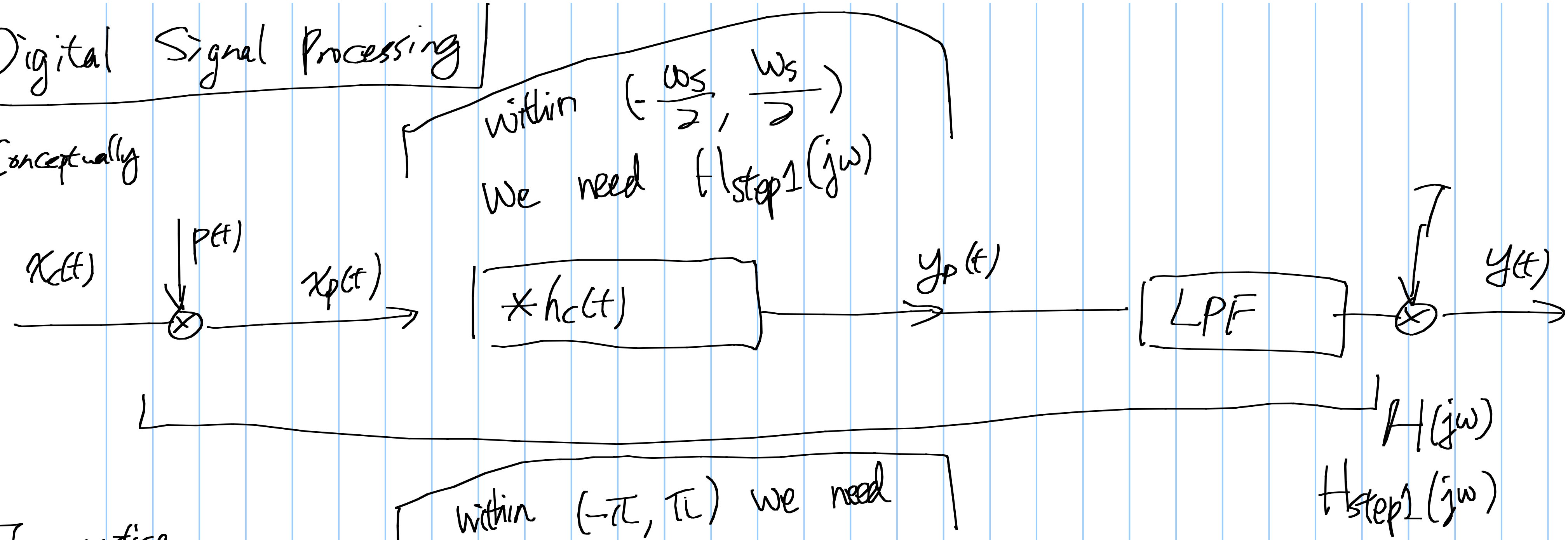
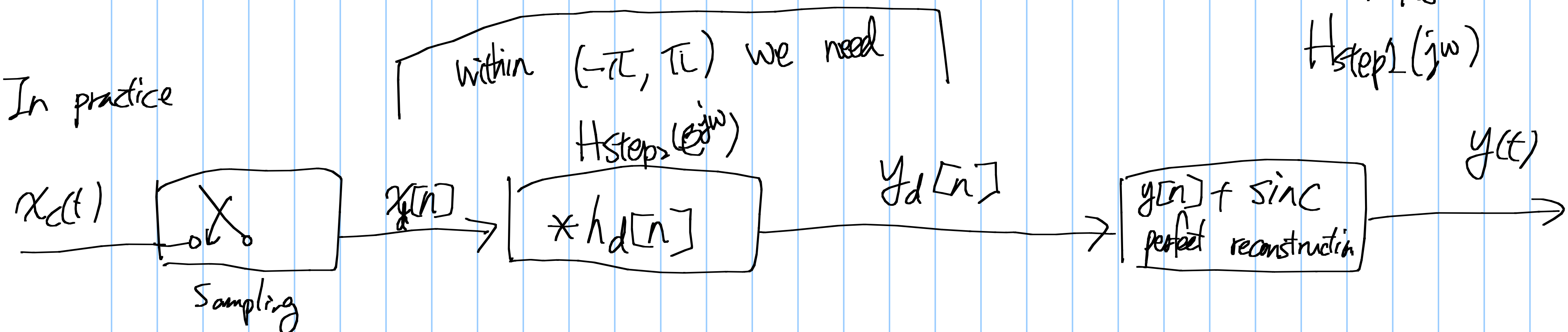
Digital Signal Processing

Conceptually



In practice



Goal: the input/output relationship between $x(t)$ & $y(t)$ fits the given design goal $h(t)$. or equivalent $H(j\omega)$

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Example: digital differentiator:

End to End

$x(t)$ $y(t)$

$$y(t) = \frac{d}{dt} x(t)$$

$\because$ $Y(j\omega) = j\omega X(j\omega)$ $\therefore$ The target $H(j\omega)$ is $H(j\omega) = j\omega = \frac{Y(j\omega)}{X(j\omega)}$

Step 1: Everything outside $\left(-\frac{\omega_s}{2}, \frac{\omega_s}{2}\right)$ does not matter.

$$H_{step1} = \begin{cases} j\omega & \text{if } |\omega| < \frac{\omega_s}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Step 2: Strech to fit $(-\pi, \pi)$

$$H_{step2}(e^{j\omega}) = H_{step1}\left(j\frac{\omega}{T}\right)$$

$$\because H_{step2}(e^{j\pi}) = H_{step1}\left(j\frac{\omega_s}{2}\right)$$

$$= H_{step1}\left(j\pi \cdot \frac{1}{T}\right)$$

$$= \begin{cases} j\frac{\omega}{T} & \text{if } |\omega| < \pi \\ 0 & \text{otherwise} \end{cases}$$

Step 3:

$$h_{step2}[n] = \mathcal{F}^{-1}\left(H_{step2}(e^{j\omega})\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(j\frac{\omega}{T}\right)e^{-j\omega n}\,d\omega$$

$$= \begin{cases} \frac{(-1)^n}{nT} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases} \quad \text{(integration by part)}$$

$$y_d[n] = x_d[n] * h_{step2}[n]$$

Q: How to compute $y_d[n]$?

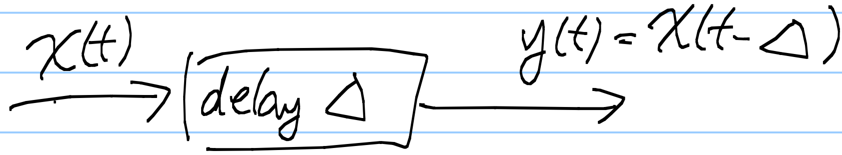
$$y[0] = \sum_{k=-\infty}^{\infty} x_d[k]\,h_{step2}[0-k]$$

$$= \cdots + x_d[-1]h_{step2}[1] + x_d[0]h_{step2}[0] + x_d[1] * h_{step2}[-1] + \cdots$$

$$= \cdots + \left(\frac{1}{2T}\right)x_d[-2] + \left(-\frac{1}{T}\right)x_d[-1] + 0 + \left(\frac{+1}{T}\right)x_d[1]$$

$$+ \left(\frac{-1}{2T}\right)x_d[2] + \cdots$$

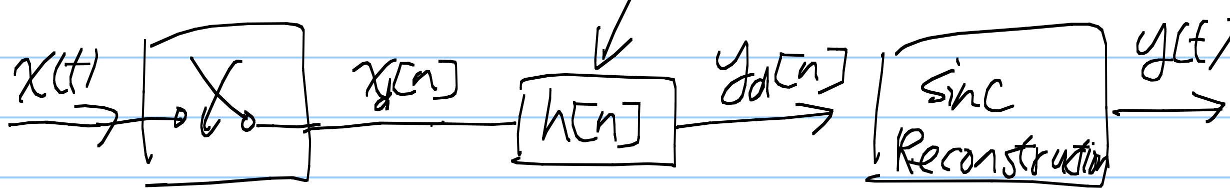
✱ Ex: CT delay component



Delay is difficult to implement in conti-time

Q: How to achieve it using sampling + DT processing

Ans: Our goal is to design



T: sampling period

to achieve the end-to-end effect

$$y(t) = x(t-\Delta)$$

If $\Delta = T$, then $y_d[n] = x_d[n-1]$

$h[n] = \delta[n-1]$ #

easy

$\Delta = 2T$ ... $y_d[n] = x_d[n-2]$

$h[n] = \delta[n-2]$

But what if $\Delta = \frac{T}{2}, \frac{T}{3}$?

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$$y(t) = x(t-\Delta) \Longleftrightarrow Y(j\omega) = \underline{\underline{e^{-j\omega\Delta}}} X(j\omega)$$

target $H(j\omega)$

Step 1: Truncate anything outside $\left(-\frac{\omega_s}{2}, \frac{\omega_s}{2}\right)$

$$\Rightarrow H_{step1}(j\omega) = \begin{cases} e^{-j\omega\Delta} & \text{if } |\omega| < \frac{\omega_s}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Step 2: Stretch $H_{step1}(j\omega)$ to fit $(-\pi, \pi)$

$$H_{step2}(e^{j\omega}) = \begin{cases} e^{-j\omega\frac{\Delta}{T}} & \text{if } |\omega| < \pi \\ \text{periodic} & \text{w. period } 2\pi \end{cases}$$

($\because$ $H_{step2}(e^{j\pi})$ must be the same as $H_{step1}\left(j\frac{\omega_s}{2}\right)$)

$$\Rightarrow \pi\Delta \times \text{const} = \frac{\omega_s}{2}\Delta$$

$$\Rightarrow \text{const} = \frac{\omega_s}{2\pi} = \frac{1}{T}$$

Step 3: Inverse DTFT

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega\frac{\Delta}{T}} e^{j\omega n}\, d\omega$$

ex: Half-sample delay $\Delta = \frac{T}{2}$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\frac{\omega}{2}} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \frac{e^{j(-\frac{1}{2}+n)\omega}\Big|_{-\pi}^{\pi}}{j(-\frac{1}{2}+n)}$$

$$= \frac{1}{2\pi} \frac{e^{j(-\frac{1}{2}+n)\pi} - e^{-j(-\frac{1}{2}+n)\pi}}{j(-\frac{1}{2}+n)}$$

$$= \frac{\sin(\pi(-\frac{1}{2}+n))}{\pi(-\frac{1}{2}+n)} = \frac{(-1)^{n+1}}{\pi(-\frac{1}{2}+n)}$$

\#

$$h[0] = \frac{-1}{\pi(-\frac{1}{2})} = \frac{2}{\pi}$$

$$h[1] = \frac{1}{\pi(\frac{1}{2})} = \frac{2}{\pi}$$

$$h[-1] = \frac{1}{\pi(-\frac{3}{2})} = \frac{-2}{3\pi}$$

$h[2]$ .. ----

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \cdots + \left(\frac{-2}{3\pi}\right) x[n+1] + \frac{2}{\pi} x[n] + \frac{2}{\pi} x[n-1] + \cdots$$

We can implement any desired conti-time delay.