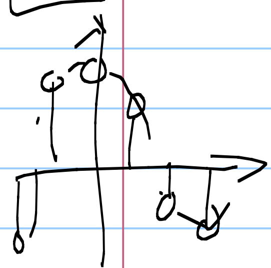
Ans: By the synthesis formula

$$h_{LPF}[n] = \frac{1}{2\pi}\int_{2\pi} H_{LPF}(e^{j\omega n})e^{j\omega n}\,d\omega$$

$\boxed{DT}$ $\boxed{CT}$



$$= \frac{1}{2\pi}\int_{-W}^{W} 1 \times e^{j\omega n}\,d\omega$$

$$= \frac{\sin Wn}{\pi n}$$

cf: $h_{LPF}(t) = \dfrac{\sin(Wt)}{\pi t}$

9. Multiplication property

$$x[n]\cdot y[n] \longleftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})\,d\theta$$

cyclic / periodic convolution $= \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})\,d\theta$

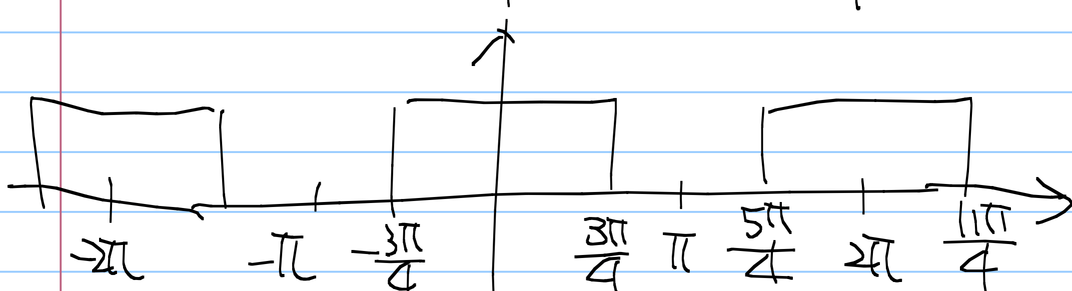
Example : 5.15

Q. $x_1[n] = \dfrac{\sin(\frac{3\pi}{4}n)}{\pi n}$, find $X_1(e^{j\omega})$.

Ans: By Example 5.12

$$X_1(e^{j\omega n}) = \begin{cases} 1 & \text{if } |\omega| < \frac{3\pi}{4} \\ 0 & \text{if } \frac{3\pi}{4} < |\omega| < \pi \end{cases}$$

& $X_1(e^{j\omega})$ is periodic

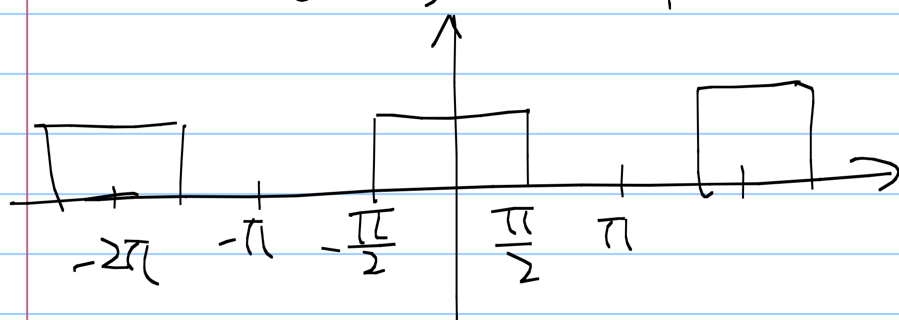


Q_2: $x_2[n] = \dfrac{\sin(\frac{\pi}{2} n)}{\pi n}$, find $X_2(e^{j\omega})$

Ans: By Example 5.12.

$$X_2(e^{j\omega}) = \begin{cases} 1 & \text{if } |\omega| < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < |\omega| < \pi \end{cases}$$

$X_2(e^{j\omega})$ is periodic.



Q_3: $y[n] = x_1[n] * x_2[n]$,

find $Y(e^{j\omega})$

Ans: By convolution property

$$Y(e^{j\omega n}) = \begin{cases} 1 & \text{if } |\omega| < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < |\omega| < \pi \end{cases}$$

$Y(e^{j\omega})$ is periodic

Q_4: $z[n] = x_1[n] \cdot x_2[n]$.

Find $Z(e^{j\omega})$.

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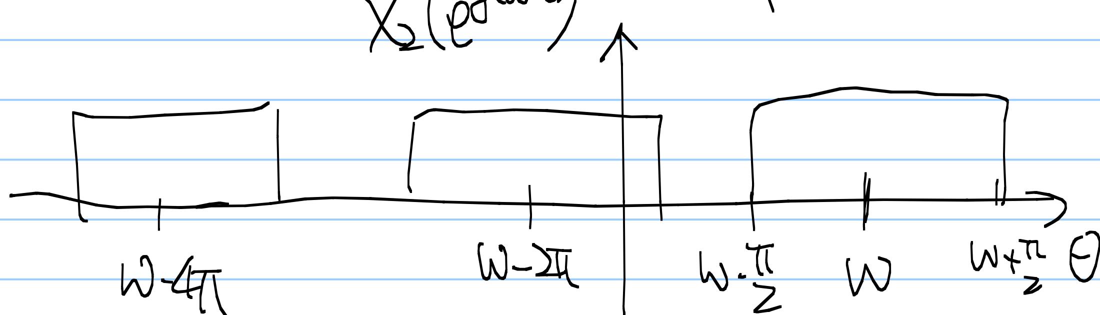
Ans: By the multiplication property

$$Z(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\cdot X_2(e^{j(\omega-\theta)})\,d\theta$$

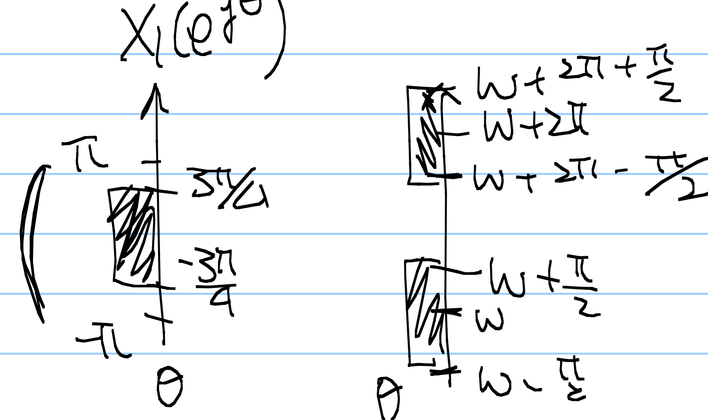
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})\cdot X_2(e^{j(\omega-\theta)})\,d\theta$$

$$X_1(e^{j\theta}) = \begin{cases} 1 & \text{if } |\theta| < \frac{3\pi}{4} \\ 0 & \text{if } \frac{3\pi}{4} < |\theta| < \pi \\ \text{periodic} & \end{cases}$$

$$X_2(e^{j(\omega-\theta)}) = \begin{cases} 1 & \text{if } -\frac{\pi}{2} < \omega-\theta < \frac{\pi}{2} \\ & \Leftarrow \omega-\frac{\pi}{2} < \theta < \omega+\frac{\pi}{2} \\ 0 & \text{if } \frac{3\pi}{4} < |\omega-\theta| < \pi. \end{cases}$$



$= \frac{1}{2\pi}\Big($



Case 1:

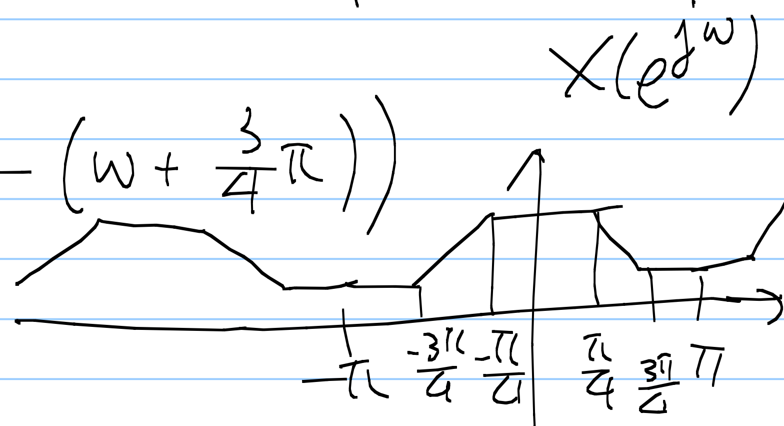
when $\begin{cases} -\pi < \omega < \pi \\ \omega + \frac{\pi}{2} > -\frac{3}{4}\pi \ \& \ (\omega + 2\pi - \frac{\pi}{2}) < \frac{3}{4}\pi \end{cases}$

$$\Rightarrow -\pi < \omega < -\frac{3\pi}{4}$$

$$= \frac{1}{2\pi}\left(\int_{-\frac{3\pi}{4}}^{\omega+\frac{\pi}{2}} 1 \cdot 1 \, d\omega + \int_{\frac{3\pi}{4}}^{2\pi+\omega-\frac{\pi}{2}} 1 \cdot 1 \right) d\theta$$

$$= \frac{1}{2\pi}\left(\omega + \frac{5\pi}{4} - \left(\omega + \frac{3}{4}\pi\right)\right)$$

$$= \frac{1}{4}$$



Case 2: when $\begin{cases} -\pi < \omega < \pi \\ \omega + \frac{\pi}{2} > -\frac{3}{4}\pi, \\ (\omega + 2\pi - \frac{\pi}{2}) > \frac{3}{4}\pi, \\ \omega - \frac{\pi}{2} < -\frac{3\pi}{4} \end{cases}$

$$\Rightarrow -\frac{3\pi}{4} < \omega < -\frac{1}{4}\pi$$

$$= \frac{1}{2\pi}\int_{-\frac{3\pi}{4}}^{\omega+\frac{\pi}{2}} 1 \cdot 1 \, d\theta = \frac{1}{2\pi}\left(\omega + \frac{5}{4}\pi\right)$$

Case 3: $\begin{cases} -\pi < \omega < \pi \\ \omega + \frac{\pi}{2} < \frac{3\pi}{4} \\ \omega - \frac{\pi}{2} > -\frac{3\pi}{4} \end{cases} \Leftrightarrow -\frac{\pi}{4} < \omega < \frac{\pi}{4}$

$$= \frac{1}{2\pi}\int_{\omega-\frac{\pi}{2}}^{\omega+\frac{\pi}{2}} 1 \, d\theta = \frac{1}{2}$$

Case 4

$$\begin{cases} -\pi < \omega < \pi \\ \omega + \frac{\pi}{2} > \frac{3\pi}{4} \\ \omega - 2\pi + \frac{\pi}{2} < -\frac{3\pi}{4} \end{cases} \iff \frac{\pi}{4} < \omega < \frac{3\pi}{4}$$

$$= \frac{1}{2\pi} \int_{\omega - \frac{\pi}{2}}^{\frac{3\pi}{4}} 1 \, d\theta$$

$$= \frac{1}{2\pi} \left(\frac{5}{4}\pi - \omega \right)$$

Case 5

$$\begin{cases} -\pi < \omega < \pi \\ \omega + \frac{\pi}{2} > \frac{3\pi}{4} \\ \omega - 2\pi + \frac{\pi}{2} > -\frac{3\pi}{4} \end{cases} \iff \frac{3\pi}{4} < \omega < \pi$$

$$= \frac{1}{2\pi} \int_{-\frac{3\pi}{4}}^{\omega - 2\pi + \frac{\pi}{2}} 1 \, d\theta + \frac{1}{2\pi} \int_{\omega - \frac{\pi}{2}}^{\frac{3\pi}{4}} 1 \, d\theta$$

$$= \frac{1}{4}$$