Properties of DTFT

Suppose $x[n] \longleftrightarrow X(e^{j\omega})$

$y[n] \longleftrightarrow Y(e^{j\omega})$

1. Linearity $\quad a\,x[n] + b\,y[n] \longleftrightarrow a\,X(e^{j\omega}) + b\,Y(e^{j\omega})$

2. Time-shift

$$y[n] = x[n-n_0] \longleftrightarrow Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

3. Freq-shift.

$$y[n] = e^{j\omega_0 n} x[n] \longleftrightarrow Y(e^{j\omega}) = X(e^{j(\omega-\omega_0)})$$

4. Time-Reversal

$$y[n] = x[-n] \longleftrightarrow Y(e^{j\omega}) = X(e^{-j\omega})$$

5. Difference in time

$$y[n] = x[n] - x[n-1]$$

$$\longleftrightarrow Y(e^{j\omega}) = (1-e^{-j\omega})X(e^{j\omega})$$

6. Differentiation in freq

$$n\,x[n] \longleftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$$

7. Parseval's relationship

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 \, d\omega$$

8. Convolution Property

$$x[n] * h[n] \longleftrightarrow X(e^{j\omega}) \cdot Y(e^{j\omega})$$

$$\begin{tikzcd}
x[n] \arrow[r, leftrightarrow, "*h[n]"] \arrow[d, leftrightarrow] & y[n] \arrow[d, leftrightarrow] \\
X(e^{j\omega}) \arrow[r, leftrightarrow, "\cdot H(e^{j\omega})"'] & Y(e^{j\omega})
\end{tikzcd}$$

Example: An LTI sys has

$h[n] = (\frac{1}{3})^n u[n]$ & input $x[n] = (\frac{1}{2})^n u[n]$

Find the output $y[n]$

Ans: $H(e^{j\omega}) = \dfrac{1}{1-(\frac{1}{3})e^{-j\omega}}$ (Table look-up)

$$X(e^{j\omega}) = \frac{1}{1-(\frac{1}{2})e^{-j\omega}}$$

$$\Rightarrow Y(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}} \cdot \frac{1}{1-(\frac{1}{2})e^{-j\omega}}$$

partial fraction

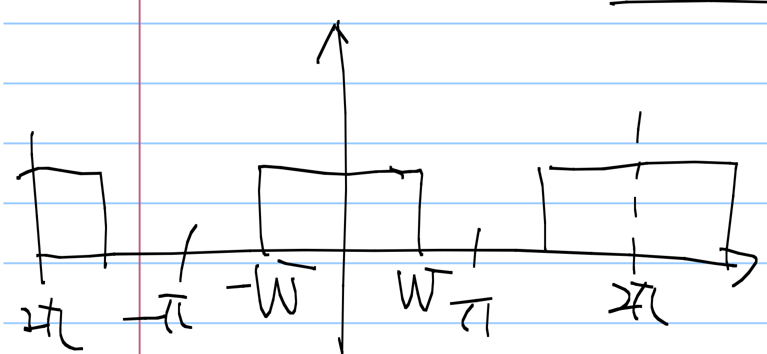
$$= \frac{-2}{1-\left(\frac{1}{3}\right)e^{-jw}} + \frac{3}{1-\left(\frac{1}{2}\right)e^{-jw}}$$

$$\Rightarrow y[n] = -2\left(\frac{1}{3}\right)^n u[n] + 3\left(\frac{1}{2}\right)^n u[n]$$

* Ideal discrete-time LPF w. cutoff freq W

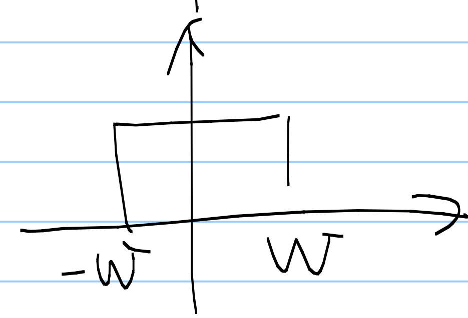
$$H(e^{jw}) = \begin{cases} 1 & \text{if } |w| < W < \pi \\ 0 & \text{if } W < |w| < \pi \end{cases}$$

& $H(e^{jw})$ is periodic



Comparison: Ideal Conti-time LPF

$$H(jw) = \begin{cases} 1 & \text{if } |w| < W \\ 0 & \text{otherwise} \end{cases}$$



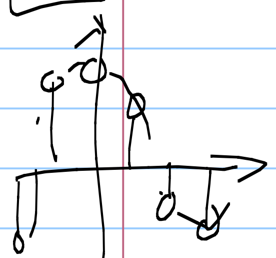
$H(jw)$ is aperiodic

Example 5.12: for a DT ideal LPF $H_{LPF}(e^{jw})$ with cutoff freq $W = \frac{2}{3}\pi$, find out the DT impulse response $h_{LPF}[n]$.

Ans: By the synthesis formula

$$h_{LPF}[n] = \frac{1}{2\pi}\int_{2\pi} H_{LPF}(e^{j\omega n}) e^{j\omega n} d\omega$$

$\boxed{DT}$



$$= \frac{1}{2\pi}\int_{-W}^{W} 1 \times e^{j\omega n} d\omega$$

$$= \frac{\sin Wn}{\pi n}$$

$\boxed{CT}$

cf: $h_{LPF}(t) = \frac{\sin(Wt)}{\pi t}$

9. Multiplication property

$$x[n]\cdot y[n] \longleftrightarrow \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$$

cyclic / periodic convolution $= \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Example : 5.15

Q1 $x_1[n] = \frac{\sin(\frac{3\pi}{4}n)}{\pi n}$, find $X_1(e^{j\omega})$.

Ans: By Example 5.12

$$X_1(e^{j\omega n}) = \begin{cases} 1 & \text{if } |\omega| < \frac{3\pi}{4} \\ 0 & \text{if } \frac{3\pi}{4} < |\omega| < \pi \end{cases}$$

& $X_1(e^{j\omega})$ is periodic