Chp 5: Discrete Time Fourier Transform (DTFT)

$\boxed{\text{DTFT}}$

Subject: Aperiodic $x[n]$

Formulas:

Synthesis $$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

Analysis $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

☆ Comparison: We use $X(e^{j\omega})$ for DTFT, $X(j\omega)$ for CTFT.

☆ $X(e^{j\omega})$ is periodic signal with respect to ω. The period is 2π.

Note Title 4/2/2010

Generalized DTFT.

subject : for periodic & aperiodic signals $x[n]$.

Ex : Suppose $x[n] = e^{j\frac{\pi}{3}n}$

Find $X(e^{j\omega})$

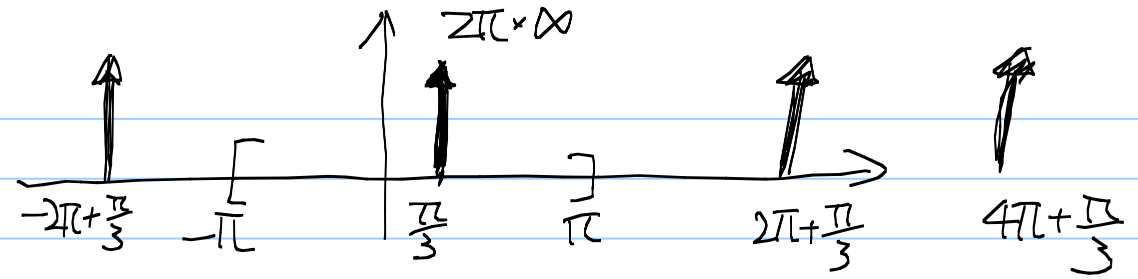
Ans : Direct Computation is for aperiodic $x[n]$. In this case, $x[n]$ has period 6. We need to use inspection

$$x[n] = e^{j\frac{\pi}{3}n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

To generate $e^{j\frac{\pi}{3}n}$

$$\Rightarrow X(e^{j\omega}) = 2\pi\, \delta(\omega - \tfrac{\pi}{3})$$



Nonetheless, we are not done yet. Since $X(e^{j\omega})$ is periodic, we must have many other impulses to make it periodic or $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - \frac{\pi}{3} - 2\pi k)$

Example: $x[n] = \cos(\frac{3}{2}\pi n)$

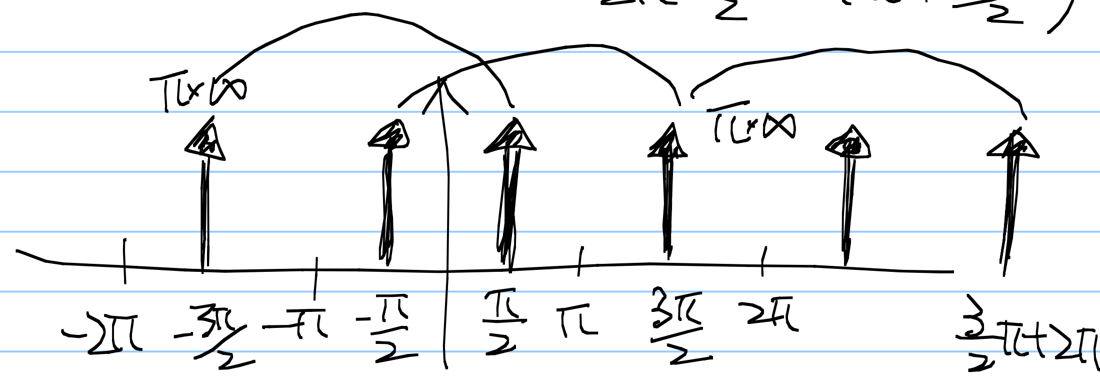
$$= \frac{1}{2} e^{j\frac{3\pi}{2}n} + \frac{1}{2} e^{-j\frac{3\pi}{2}n}$$

Find $X(e^{j\omega})$.

Ans: By inspection

Step 1: $X(e^{j\omega}) = 2\pi \times \frac{1}{2}\, \delta(\omega - \frac{3\pi}{2})$

$$+ 2\pi \times \frac{1}{2}\, \delta(\omega + \frac{3\pi}{2})$$



Step 2: Make it periodic

$$X(e^{j\omega}) = \begin{cases} \pi\,\delta(\omega - \frac{\pi}{2}) + \pi\,\delta(\omega + \frac{\pi}{2}) & \forall\ |\omega| < 2\pi \\ \text{periodic w. } 2\pi \end{cases}$$

Final Answer: $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \pi\,\delta(\omega - \frac{\pi}{2} - 2k\pi) + \pi\,\delta(\omega + \frac{\pi}{2} - 2k\pi)$

Ex $x[n] = \sin(\frac{5}{4}\pi n)$

Find & plot $X(e^{j\omega})$

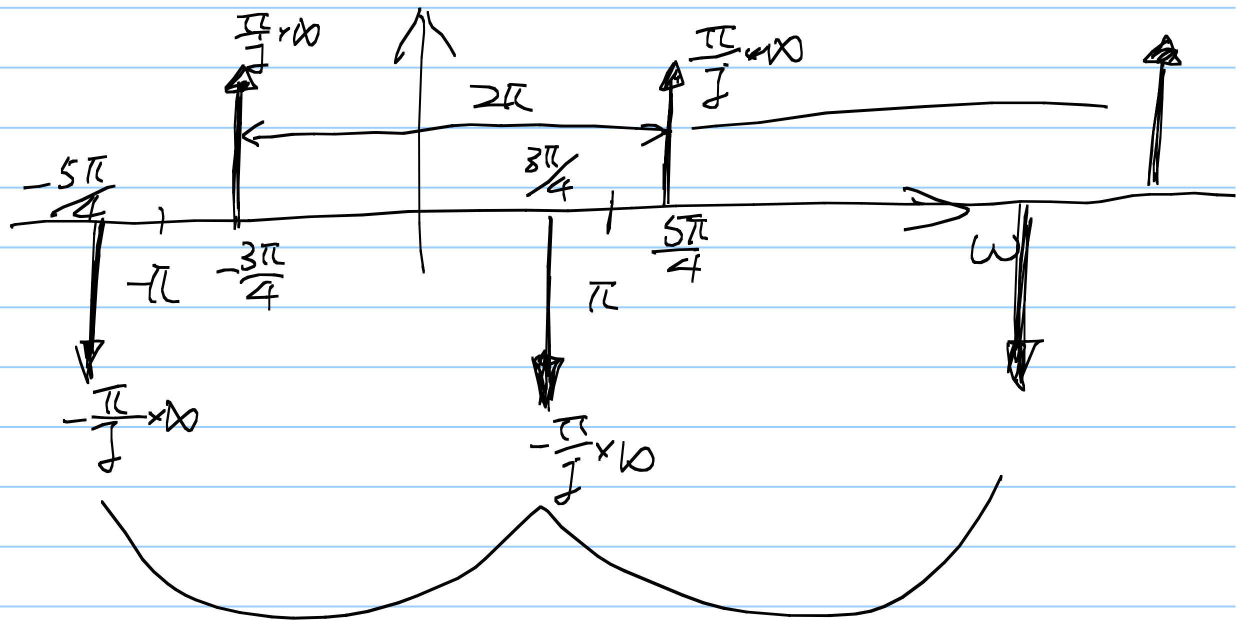
Ans: By inspection.

$$x[n] = \frac{1}{2j} e^{j\frac{5\pi}{4}n} - \frac{1}{2j} e^{-j\frac{5}{4}\pi n}$$

Step 1

$$X(e^{j\omega}) = 2\pi \times \frac{1}{2j}\,\delta(\omega - \frac{5\pi}{4}) - 2\pi \times \frac{1}{2j} \times \delta(\omega + \frac{5}{4}\pi)$$

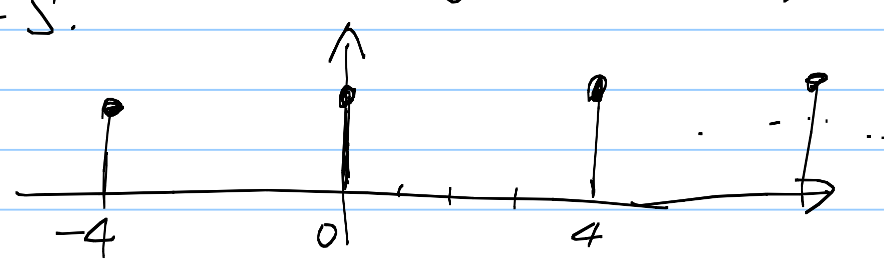
Step 2: Make it periodic



$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} -\frac{\pi}{j}\,\delta(\omega - \frac{3\pi}{4} - 2k\pi) + \frac{\pi}{j}\,\delta(\omega + \frac{3\pi}{4} - 2k\pi)$$

* For general periodic signals, we rely on DTFS.

Ex: $x[n] = \sum_{k=-\infty}^{\infty} \delta(n-4k)$



(Example 5.6)
Find $X(e^{j\omega})$

Ans: Now we are facing a DT periodic signal that cannot be directly solved by inspection.

Step 0: Find its DTFS.

$$a_k = \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{N} \times (1 \cdot 1 + 0) = \frac{1}{N}$$

$$\Rightarrow x[n] = \sum_{k=0}^{N-1} \frac{1}{N} e^{jk\frac{2\pi}{N} \times n}$$

Step 1: By inspection,

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=0}^{N-1} \delta\left(\omega - k\frac{2\pi}{N}\right)$$

Step 2: Make it periodic

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{m=-\infty}^{\infty} \sum_{k=0}^{N-1} \delta\left(\omega - k\frac{2\pi}{N} - 2\pi m\right)$$

$$= \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{N}\right) \quad \#$$

Plot: