* Generalized CTFT

Subject: $x(t)$ can be periodic or aperiodic

Example: $x(t) = \cos(\frac{2\pi}{3}t)$

Q: The F.S of $x(t)$.

Ans: ① $\omega_0 = \frac{2\pi}{3}$

② $x(t) = \frac{1}{2} e^{j\frac{2\pi}{3}t} + \frac{1}{2} e^{j(-1)\frac{2\pi}{3}t}$ ——— ①

$\Rightarrow a_1 = a_{-1} = \frac{1}{2}$, all other $a_k = 0$.

Q: Can we generalize CT FT so that we can use CTFT to represent $x(t)$ as well?

Ans: Compare ① with the synthesis formula

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

What if $X(j\omega) = \alpha\,\delta(\omega - \omega_0)$?

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \alpha\,\delta(\omega - \omega_0) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \alpha\, e^{j\omega_0 t}$$

By comparison, if we choose

$$X(j\omega) = 2\pi a_1 \delta(\omega - (1)\omega_0)$$
$$+ 2\pi a_2 \delta(\omega - (-1)\cdot\omega_0)$$
$$= \pi\delta(\omega - \frac{2\pi}{3}) + \pi\delta(\omega + \frac{2\pi}{3})$$

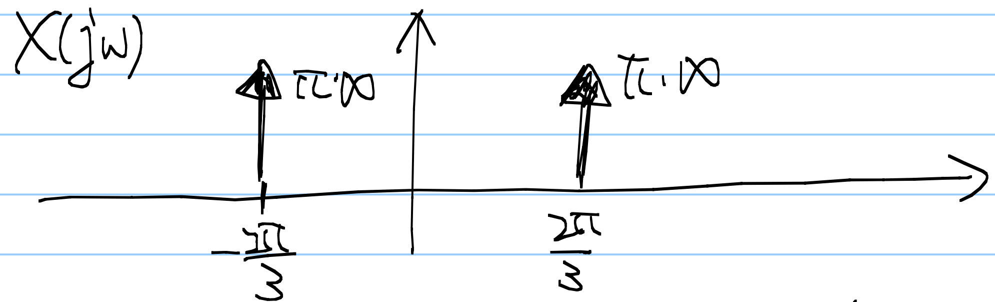
then $x(t) = \frac{1}{2\pi}\cdot 2\pi a_1 e^{j1\cdot\omega_0 t} + \frac{1}{2\pi} 2\pi a_{-1} e^{j(-1)\omega_0 t}$

$$= a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$$

$$= \cos(\frac{2\pi}{3} t)$$

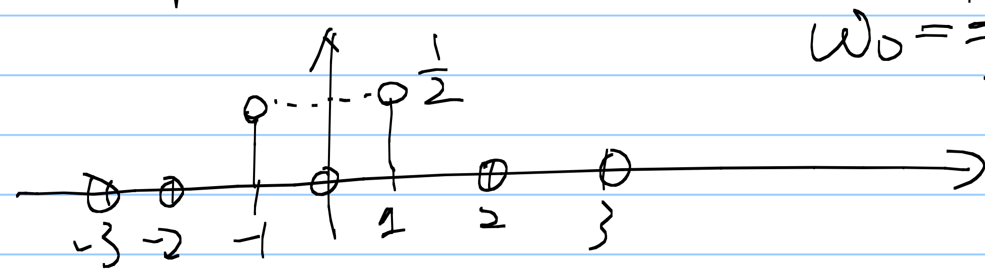
$$\Rightarrow \cos(\frac{2\pi}{3}t) \xleftrightarrow{F.T} \pi\delta(\omega - \frac{2\pi}{3}) + \pi\delta(\omega + \frac{2\pi}{3})$$

* How to plot $X(j\omega)$? A CT signal



* For comparison: How to plot a_k?

$\omega_0 = \frac{2\pi}{3}$



* The above inspection method can be applied even to other complex exponentials.

$\boxed{\text{Very similar to HW9 Q77}}$

Example: $x(t) = \cos(t) + \sin(\pi t)$

Q1: Is $x(t)$ periodic?

Ans: $\because$ $LCM\left(\frac{2\pi}{1}, \frac{2\pi}{\pi}\right)$ does not exist.

$\therefore$ Not periodic

Q2: Find $X(j\omega)$

Ans: $$x(t) = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt} + \frac{1}{2j}e^{j\pi t} - \frac{1}{2j}e^{-j\pi t}$$

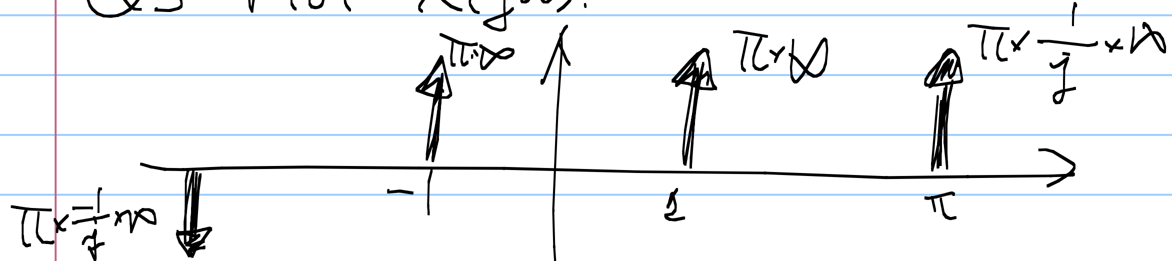
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

By inspection

$$\frac{1}{2\pi}e^{jat} \longleftrightarrow \delta(\omega - a)$$

$$\Rightarrow X(j\omega) = 2\pi \times \frac{1}{2}\,\delta(\omega - 1) + 2\pi \times \frac{1}{2}\,\delta(\omega - (-1)) + 2\pi \times \frac{1}{2j}\,\delta(\omega - \pi) + 2\pi \times \left(\frac{-1}{2j}\right)\delta(\omega - (-\pi))$$

Q3: Plot $X(j\omega)$.



* For general periodic $x(t)$, how to find its F.T.

Ans: Step 1: Find the CTFS of $x(t)$

① ω_0

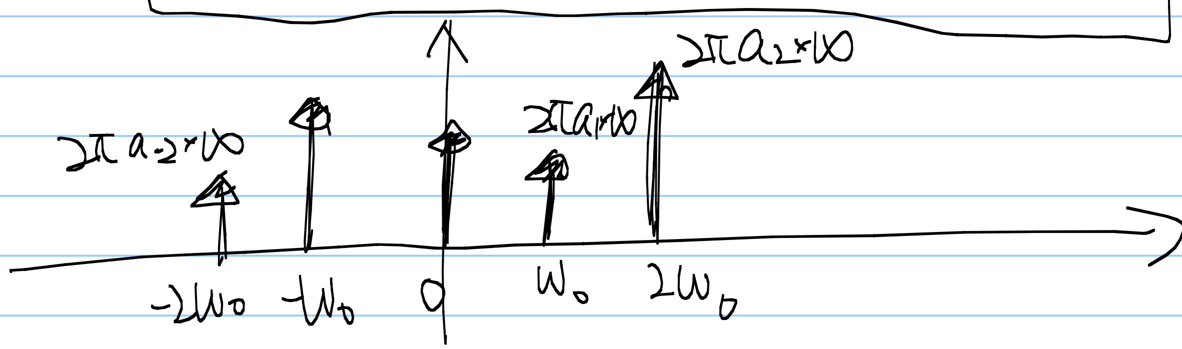
② a_k

Step 2: $\because x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

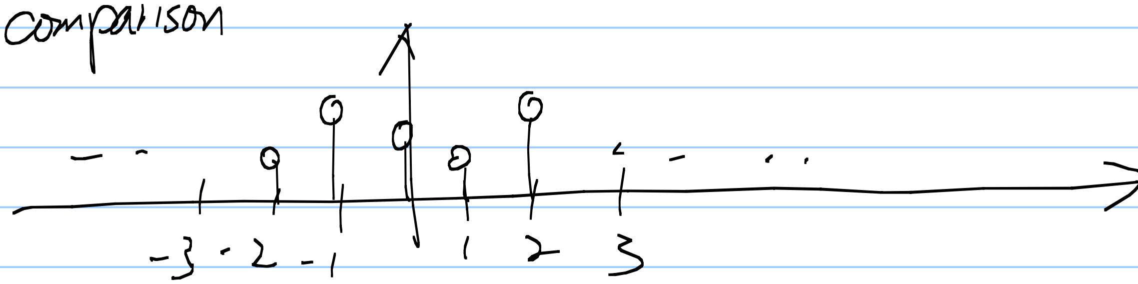
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$\therefore$ By inspection

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{\infty} (2\pi a_k)\, \delta(\omega - k\omega_0)}$$



For comparison

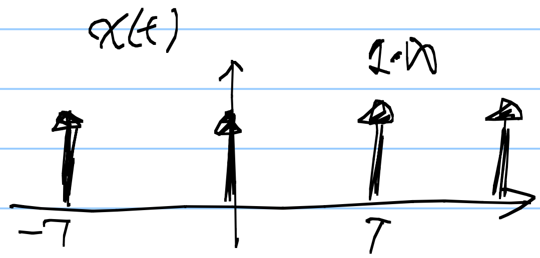


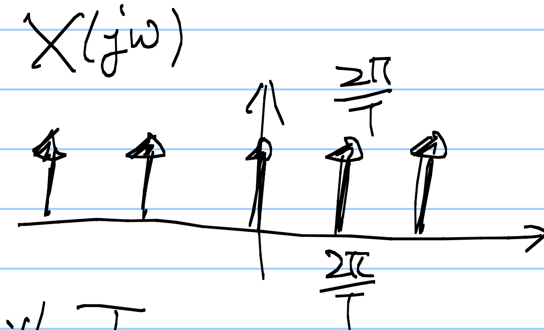
The FT kills two birds in one stone: Both ω_0 & the height.

* Text Example 4.8

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \qquad \text{Plot } x(t)$$

Find $X(j\omega) = \boxed{\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T}),}$ plot $X(j\omega)$





pf: $x(t)$ is periodic with T

$\Rightarrow$ We need generalized FT

Step 1: F.S of $x(t)$

① T

② $a_k = \frac{1}{T} \int x(t) e^{-jk\frac{2\pi}{T}t} dt$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\frac{2\pi}{T}t} dt$$

$$= \frac{1}{T} e^{-jk\frac{2\pi}{T}0} = \frac{1}{T}$$

Step 2: $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\frac{2\pi}{T})$

$$= \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T})$$

* Impulse trains in time $\Leftrightarrow$ Impulse trains in freq