Note Title 3/7/2014

* Chapter 4 CT Fourier Transform

subject : CT aperiodic $x(t)$

Representation :

$\boxed{\text{CTFT Synthesis formula}}$

$$x(t) = \int_{\omega=-\infty}^{\infty} \underbrace{a_\omega}_{\text{coeff}} \underbrace{e^{j\omega t}}_{\text{signal}} d\omega$$

$\boxed{\text{CTFT Analysis formula}}$

$$a_\omega = \frac{1}{2\pi} \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Comparison to CTFS.

CTFS : $\begin{cases} x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \end{cases}$

The above equation represents $x(t)$ by the integral of test HRCE signals $e^{j\omega t}$ for different ω values.

Unfortunately, it is not the most common form in the FT literature. We need some modifications.

Modification:

Define: $X(j\omega) = a_\omega \cdot 2\pi \Rightarrow a_\omega = \frac{X(j\omega)}{2\pi}$

$$\Rightarrow \boxed{x(t) = \int_{\omega=-\infty}^{\infty} \frac{X(j\omega)}{2\pi} e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega}$$

$$\& \quad \boxed{X(j\omega) = 2\pi \cdot a_\omega = 2\pi \cdot \frac{1}{2\pi}\int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

* We say $X(j\omega)$ is the Fourier transform of $x(t)$. $x(t)$ is the inverse Fourier transform of $X(j\omega)$

Jointly $(x(t), X(j\omega))$ form a F.T. pair.

* We sometimes write

$$X(j\omega) = \mathcal{F}(x(t))$$

$$x(t) = \mathcal{F}^{-1}(X(j\omega))$$

Example: Text Example 4.2

$x(t) = e^{-a|t|}$ for some $a>0$

Find its FT $X(j\omega)$

Ans: By Direct computation

$$X(j\omega) = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{t=-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt$$

$$= \int_{t=-\infty}^{0} e^{-a(-t)} e^{-j\omega t}\, dt + \int_{t=0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{2a}{a^2+\omega^2}$$

Example: $X(j\omega) = e^{-b|\omega|}$ for some $b>0$. Find $x(t)$

Ans: Direct computation

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left(\int_{\omega=-\infty}^{0} e^{-b(-\omega)} e^{j\omega t}\, d\omega + \int_{\omega=0}^{\infty} e^{-b\omega} e^{j\omega t}\, d\omega \right)$$

$$= \frac{1}{2\pi}\left(\frac{1}{b-jt} + \frac{1}{b+jt} \right)$$

$$= \frac{1}{\pi} \times \frac{b}{b^2+t^2}$$

* The formulas:

$$\begin{cases} x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \\ X(j\omega) = \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt \end{cases}$$

* The two formulas are VERY SIMILAR but not identical. This is good in the sense that many computation can be reused. But this is also bad in the sense that it is very confusing for the first-time users.

* A side note: You might have noticed that we have seen this formula before

CT periodic

$$\begin{tikzcd} x(t) \arrow[rr, leftrightarrow, "*h(t)"] \arrow[d, leftrightarrow, "F.S."'] & & y(t) \arrow[d, "F.S."] \\ a_k, \omega_0 \arrow[rr, "H(j\omega)"'] & & b_k = a_k \cdot H(jk\omega_0) \end{tikzcd}$$

F.T.

ω_0

$$H(j\omega) = \int_{t=-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

is indeed the F.T. of the impulse response

We term $H(j\omega)$, the <u>freq response</u>.

* The freq response $H(j\omega)$ is the FT of the impulse response $h(t)$.