* Let us revisit the CT FS.

Property ⑧ for CT FS.

$x(t) \longleftrightarrow a_k, \omega_0.$

$y(t) = x(t) * h(t)$ is the output when passing $x(t)$ through a LTI system $h(t)$

$$y(t) \longleftrightarrow \boxed{b_k = a_k \cdot H(jk\omega_0)}, \omega_0$$

where $$H(j\omega) = \int_{s=-\infty}^{\infty} h(s)\, e^{-j\omega s}\, ds$$

$$H(jk\omega_0) = \int_{s=-\infty}^{\infty} h(s)\, e^{-jk\omega_0 s}\, ds$$

pf: $$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

LTI system $\quad e^{j\omega t} \xrightarrow{h(t)} H(j\omega)\, e^{j\omega t}$

$$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k \cdot H(jk\omega_0)}_{\text{must be } b_k} e^{jk\omega_0 t}$$

* This is the big picture of learning FS

$$\begin{tikzcd}
x(t) \arrow[rr, leftrightarrow, "* h(t)"] \arrow[d, leftrightarrow, "F.S"'] & & y(t) \arrow[d, leftrightarrow, "F.S"] \\
a_k \arrow[rr, leftrightarrow, "H(j\omega) = \int_{-\infty}^{\infty} h(s) e^{-j\omega s} ds"] & & b_k = a_k \cdot H(jk\omega_0)
\end{tikzcd}$$

There are two ways of computing the FS of $y(t)$

Route 1

$x(t) \xrightarrow{h(t)} y(t) = h(t) * x(t)$

$\downarrow$ by ① inspection / ② direct computation, ③ properties

(b_k, w_0)

Route 2

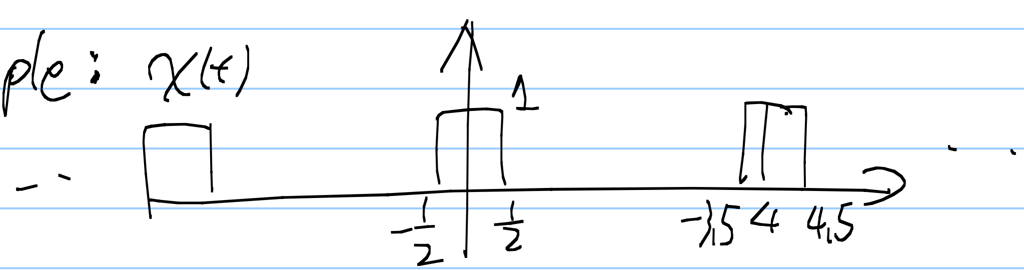
$x(t)$

$\downarrow$ F.S.

$(a_k, w_0) \xrightarrow{H(jw)} a_k = b_k H(jkw_0)$

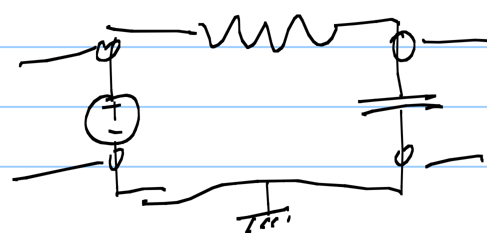
w_0

Example: $x(t)$



$x(t) \rightarrow \boxed{h(t) = e^{-t}u(t)} \rightarrow y(t)$

$x(t)$ rectangular waveform



$y(t)$

Q: Find the F.S of $y(t)$

Ans: Step 1: Find the F.S of $x(t)$ first

$$a_0 = \frac{2\times\frac{1}{2}}{4} = \frac{1}{4}$$

$$a_k = \frac{\sin\left(k\times\frac{2\pi}{4}\times\frac{1}{2}\right)}{k\pi} = \frac{\sin\left(k\frac{\pi}{4}\right)}{k\pi}$$

Step 2: Compute $H(j\omega)$ & $H(jk\omega_0)$

$$H(j\omega) = \int_{-\infty}^{\infty} h(s)\, e^{-j\omega s}\, ds = \int_{0}^{\infty} e^{-s} e^{-j\omega s}\, ds = \frac{1}{1+j\omega}$$

$$H(jk\omega_0) = \frac{1}{1+jk\frac{2\pi}{4}} = \frac{1}{1+jk\pi/2}$$

Step 3: $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$$b_k = a_k \frac{1}{1+j\frac{k\pi}{2}}$$

$$b_0 = a_0 \times 1 = \frac{1}{4}$$

$\forall k \neq 0$ $\quad b_k = \frac{\sin\left(k\frac{\pi}{4}\right)}{k\pi} \times \frac{1}{1+jk\frac{\pi}{2}}$ ※

Different freq components exhibit different gain $H(j\omega)$

Similarly for DTFS.

$$
\begin{tikzcd}
x[n] \arrow[rr, leftrightarrow, "*h[n]"] \arrow[d, leftrightarrow] & & y[n] = x[n] * h[n] \arrow[d] \\
a_k,\ \omega_0 \arrow[rr, "H(e^{j\omega})"] & & \begin{cases} b_k = a_k H(e^{jk\omega_0}) \\ \omega_0 \end{cases}
\end{tikzcd}
$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

* Pay attention to the upper/lower limits of the summation

Q: In practice, how to decide the freq characteristics of an unknown LTI sys?

Ans: Step 1: Record $h(t)$ by sending $\delta(t)$ as input

Step 2: Compute $H(j\omega) = \int_{s=-\infty}^{\infty} h(s) e^{-j\omega s} ds$

Step 3: Evaluate $|H(j\omega)|^2$ as a function of ω.