* Convergence of the Fourier Series.

Q: Do all CT periodic signals x(t) have a Fourier series representation, i.e. can every periodic signal be written as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \;?$$

A: The answer is no. However, we can get close in most cases.

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* We can find the Fourier series representation for the following types of signals:

(1) Any "continous" signal x(t) that has no abrupt changes can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

(2) If a "mostly continuous" signal x(t) does not go to infinity and has a finite number of abrupt points in a single period T, then only at those points, we have

$$x(t) \neq \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

For all those "continuous" regions, the FS representation holds.

* See Conditions 1 to 3 in p. 197 of the textbook for detailed discussion.

* Properties of the CT FS.

Note Title 9/25/2012

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{+jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$

That is

$$x(t) \xleftrightarrow{FS} (a_k, \omega_0)$$

time-domain representation — freq-domain representation

Given $x(t)$, we find a_k by ① inspection or ② direct computation.

Suppose $x_2(t) = x(t-t_0)$ is a shifted version of the original $x(t)$ & we have spent a lot of time to compute a_k

$$x(t) \xrightarrow[\text{② direct computation}]{\text{① inspection}} a_k, \omega_0$$

③ The properties of FS.

$$x_2(t) = x(t-t_0) \longleftrightarrow b_k, \omega$$

Q: Can we directly compute b_k from a_k w/o reapplying ① or ②

A: Yes, by the properties of F.S,

* Properties of FS. (see Table 3.1 for a complete list of properties.

① Linearity: Suppose both $x(t)$ & $y(t)$ have period T.

$$x(t) \longleftrightarrow a_k, \quad \boxed{\omega_0 = \frac{2\pi}{T}}$$

$$y(t) \longleftrightarrow b_k, \quad \boxed{\omega_0 = \frac{2\pi}{T}}$$

must be the same.

$$z(t) = A x(t) + B y(t)$$

$$z(t) \longleftrightarrow c_k, \quad \boxed{\omega_0}$$

$$\Rightarrow \underline{\quad} = \omega_0, \quad \boxed{c_k = A a_k + B b_k}$$

pf: ① Direct computation

$$c_k = \frac{1}{T}\int_T \left(A x(t) + B y(t)\right) e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T A x(t) e^{-jk\omega_0 t} + \frac{1}{T}\int_T B y(t) e^{-jk\omega_0 t}\, dt$$

$$= A a_k + B b_k.$$

$$z(t) = A\left(\sum_{k=-\infty}^{\infty} a_k e^{+jk\omega_0 t}\right) + B\left(\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}\right)$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{(Aa_k + Bb_k)}_{\text{must be } c_k} e^{jk\omega_0 t}$$

② Time-Shift property

$$x(t) \xleftrightarrow{F.S} (a_k, \omega_0)$$

$$y(t) = x(t-t_0) \xleftrightarrow{F.S} (b_k, \boxed{\omega_0})$$

Ans: ____ $= \omega_0$ $\because$ time-shift does not change the period.

$$\boxed{b_k = a_k \underbrace{e^{-jk\omega_0 t_0}}}$$

It does not depend on t.

pf ①: Direct computation — p.202

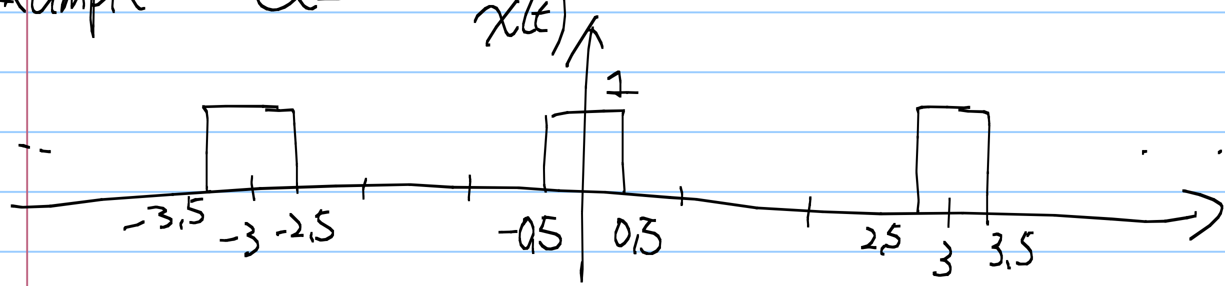
②: By inspection

$$y(t) = x(t-t_0)$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t-t_0)}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{a_k \cdot e^{-jk\omega_0 t_0}}_{\text{must be } b_k} \cdot e^{jk\omega_0 t}$$

Example Q1



Find the FS of $x(t)$

① $w_0 = \frac{2\pi}{3}$ #

Ans: ② By the formula in Example 3.5

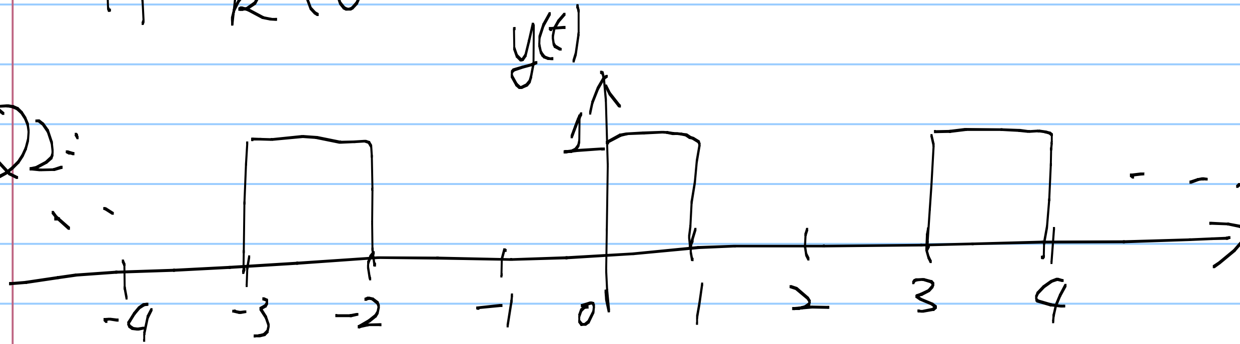
Plug in $T=3$ $T_1 = \frac{1}{2}$ $w_0 = \frac{2\pi}{3}$

$$a_0 = \frac{2T_1}{T} = \frac{1}{3}$$

$$a_k = \frac{\sin(kw_0T_1)}{k\pi} = \frac{\sin(k \times \frac{2\pi}{3} \times \frac{1}{2})}{k\pi} = \frac{\sin(\frac{k\pi}{3})}{k\pi}$$

if $k \neq 0$

Q2:



Find the FS of $y(t)$

Ans: ① $w_0 = \frac{2\pi}{3}$

② by the time-shift property

$$b_k = a_k e^{-jkw_0 \underline{t_0}}$$

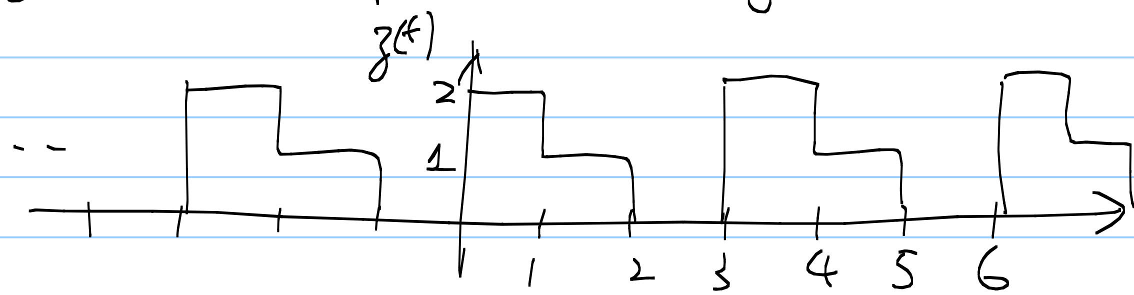
time-shift

$$\Rightarrow b_0 = a_0 \times e^{j0 \times \frac{2\pi}{3} \times \frac{1}{2}} = a_0 = \left(\frac{1}{3}\right)$$

$$b_k = a_k \times e^{-jk\frac{2\pi}{3} \times \frac{1}{2}}$$

$$= \frac{\sin\left(\frac{k\pi}{3}\right)}{k\pi} \times e^{-j\frac{k\pi}{3}}$$

Q3: HW6Q57: Prob 3.22(a) fig (f)



Find the FS of $z(t)$

Ans: ① $\omega_0 = \frac{2\pi}{3}$

② $z(t) = 2y(t) + y(t-1)$

$$\Rightarrow c_k = 2 \times b_k + b_k \times e^{-jk\omega_0 \times \underline{1}}$$ → time-shift

$$\Rightarrow c_0 = 2 \times \left(\frac{1}{3}\right) + 1 \times \frac{1}{3} = 1.$$

for $k \neq 0$

$$c_k = \frac{\sin\left(\frac{k\pi}{3}\right)}{k\pi} \times e^{-j\frac{k\pi}{3}}\left(2 + e^{-j\frac{2k\pi}{3}}\right)$$