

CT periodic signals

Print the FS convergence
handout

Consider a CT $x(t)$ with period T .
Its freq is $\omega_0 = \frac{2\pi}{T}$

Our goal is to write

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k}_{\text{coeff}} \underbrace{e^{jk\omega_0 t}}_{\text{test signals: HRCE}}$$

We have ∞ many of
HRCEs.

This representation is called the
Fourier series representation

Q: Why called Fourier? Name after a French
mathematician

Q: Why series? $\because x(t) = \sum_{k=-\infty}^{\infty} (\dots)$

Q: Why "representation"? $x(t)$ is equivalently
expressed by the series

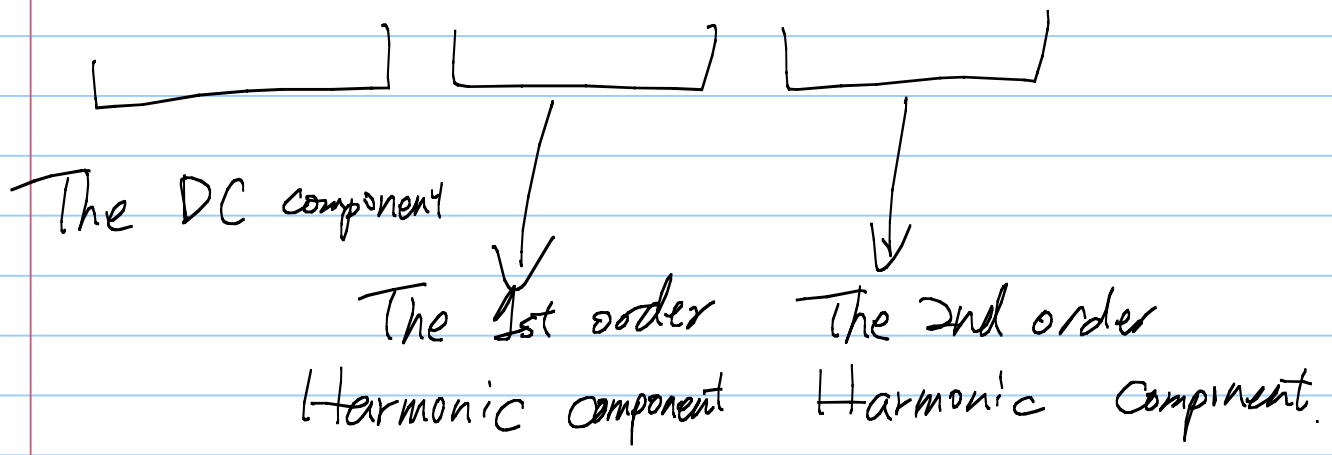
Other representation example:

Taylor's expansion: $\frac{1}{1-x} = 1 + x + x^2 + \dots$ for $|x| < 1$.

* The complex numbers $a_0, a_{-1}, a_1, a_{-2}, a_2, \dots$ are called the Fourier series coefficients.

$$X(t) = a_0 \cdot e^{j0 \cdot \omega_0 t} + a_{-1} e^{j(-1) \omega_0 t} + a_1 e^{j(1) \omega_0 t} + a_{-2} e^{j(-2) \omega_0 t} + a_2 e^{j(2) \omega_0 t} + \dots$$

$e^0 = 1$



Q: How to compute a_k ?

A: We have a formula:

$$a_k = \frac{1}{T} \int_0^T X(t) e^{-jk\omega_0 t} dt$$

a value indexed by k .

Note that since both $X(t)$ and $e^{-jk\omega_0 t}$ have period T , sometimes it is easier if we integrate $\frac{1}{T} \int_1^{1+T} X(t) e^{-jk\omega_0 t} dt$ instead

For simplicity, we use \int_T to denote "integrating over any single period"

$$a_k = \int_T X(t) e^{-jk\omega_0 t} dt$$

* Derivation of the formula

$$\begin{aligned}
 & \int_0^T x(t) e^{-jk\omega_0 t} dt \\
 &= \int_0^T \left(\sum_{h=-\infty}^{\infty} a_h e^{jh\omega_0 t} \right) e^{-jk\omega_0 t} dt \\
 &= \int_0^T \sum_{h=-\infty}^{\infty} a_h e^{j(h-k)\omega_0 t} dt \\
 &= \sum_{h=-\infty}^{\infty} a_h \underbrace{\int_0^T e^{j(h-k)\omega_0 t} dt}_{= \begin{cases} T & \text{if } h=k \\ 0 & \text{otherwise} \end{cases}} \\
 &= a_k \cdot T
 \end{aligned}$$

$$\Rightarrow a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt.$$

Summary

$$\boxed{a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt} : \text{FS analysis equation}$$

Analysis of $x(t)$ by finding the coeff. of the harmonic components.

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}} : \text{FS synthesis equation}$$

Synthesis of $x(t)$ from the weighted sum of the harmonic components.