CT periodic signals

Print the FS convergence handout

Consider a CT $x(t)$ with period T.
Its freq is $\omega_0 = \frac{2\pi}{T}$

Our goal is to write

$$x(t) = \sum_{k=-\infty}^{\infty} \underline{\underline{a_k}}\ \underline{\underline{e^{jk\omega_0 t}}}$$

↳ coeff

↳ test signals: HRCE
We have ∞ many of HRCEs.

This representation is called the
Fourier series representation

Q: Why called Fourier? Name after a French mathematician

Q: Why series? $\because$ $x(t) = \sum_{k=-\infty}^{\infty} (\cdots\cdots)$

Q: Why "representation"? $x(t)$ is equivalently expressed by the series

Other representation example:

Taylor's expansion: $\frac{1}{1-x} = 1 + x + x^2 + \cdots\cdots$ for $|x|<1$.

* The complex numbers $a_0,\ a_{+1}\ a_2\ \cdots$
$a_{-1}\ a_{-2}\ \cdots$
are called the Fourier series coefficients.

$$x(t) = \underbrace{a_0 \cdot e^{j0\cdot w_0 t}}_{\text{The DC component}} + \underbrace{a_1 e^{j1\times w_0 t} + a_{-1} e^{j(-1)w_0 t}}_{\text{The 1st order Harmonic component}} + \underbrace{a_2 e^{j2w_0 t} + a_{-2} e^{j(-2)w_0 t}}_{\text{The 2nd order Harmonic compinent}} + \cdots$$

$e^0 = 1$

Q: How to compute a_k ?

A: We have a formula:

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jkw_0 t}\, dt$$

a value indexed by k.

Note that since both $x(t)$ and $e^{-jkw_0 t}$ have period T, sometimes it is easier if we integrate

$$\frac{1}{T}\int_1^{1+T} x(t)\, e^{-jkw_0 t}\, dt$$

instead

For simplicity, we use $\int_T$ to denote "integrating over any single period"

☆☆☆ $\boxed{a_k = \int_T x(t)\, e^{-jkw_0 t}\, dt}$

* Derivation of the formula

$$\int_0^T x(t)e^{-jk\omega_0 t}\,dt$$

$$=\int_0^T\left(\sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}\right)e^{-jk\omega_0 t}\,dt$$

$$=\int_0^T \sum_{n=-\infty}^{\infty} a_n e^{j(n-k)\omega_0 t}\,dt$$

$$=\sum_{n=-\infty}^{\infty} a_n \underline{\underline{\int_0^T e^{j(n-k)\omega_0 t}\,dt}}$$

$$=\begin{cases} T & \text{if } n=k \\ 0 & \text{otherwise} \end{cases}$$

$$= a_k \cdot T$$

$$\Rightarrow a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt.$$

Summary

$\boxed{a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt}$: FS analysis equation

Analysis of $x(t)$ by finding the coeff. of the harmonic components.

$\boxed{x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$: FS synthesis equation

Sythesis of $x(t)$ from the weighted sum of the harmonic components.