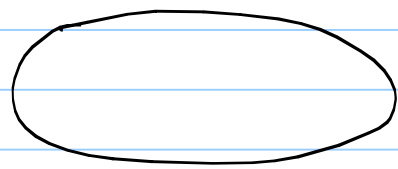
⑥ Linearity vs. Non-linearity. See Lectures 1-3.

Signals.



Systems

Why LTI systems?

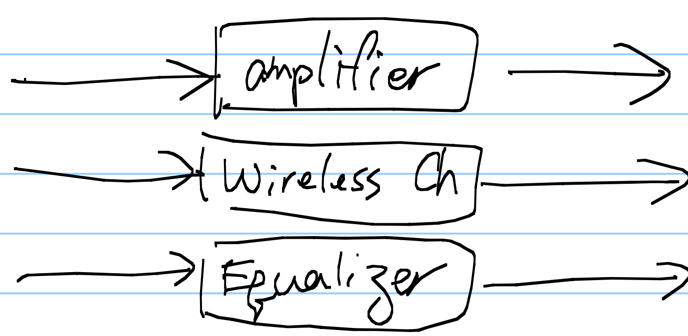
① The analysis is simple & elegant.

② A lot of systems can be well approximated by a LTI system.

Linear: ① double the input, the output is also doubled
② Combine two inputs, the outputs are also combined.

Time-Invariant: When do we apply the input signal does not matter.

Ex:



$\boxed{\text{DT-LTI}}$ Use the shifted impulses as the test signals

$$x_k[n] = \delta[n-k]$$

$x_1[n] \quad \cdots\cdots \quad x_k[n]$

$y_1[n] \quad \quad y_k[n]$

We know

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{coeff}} \ \underbrace{\delta[n-k]}_{\text{test signal}}$$

coeff / corresponding output

$$\Longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]\, y_k[n]$$

Notice that if the input is $\delta[n]$

$$\delta[n] = x_0[n] \longrightarrow \boxed{\text{LTI}} \xrightarrow{y_0[n]}$$

We thus say $y_0[n]$ is the (unit) impulse response of the system. For convenience, we denote $h[n] \triangleq y_0[n]$

Def: An impulse response $h[n]$ is the output when the input is an impulse. (S08, MT1Q1)

Question: can we express $y_1[n]$ by $y_0[n] = h[n]$?

Ans: Yes. by $\boxed{\text{Time-Invariance}}$

$$x_1[n] = \delta[n-1] \longrightarrow \boxed{LTI} \longrightarrow y_1[n] = y_0[n-1] = h[n-1]$$

A general form of $y_k[n]$ is thus

$y_k[n] = h[n-k]$, the shifted version of the impulse response

$x_k[n] = \delta[n-k]$ the shifted version of the impulse

$\downarrow$ LTI

$y_k[n] = h[n-k]$ the shifted version of the impulse response

$$\Rightarrow \boxed{y[n] = \sum_{k=-\infty}^{\infty} \alpha_k y_k[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

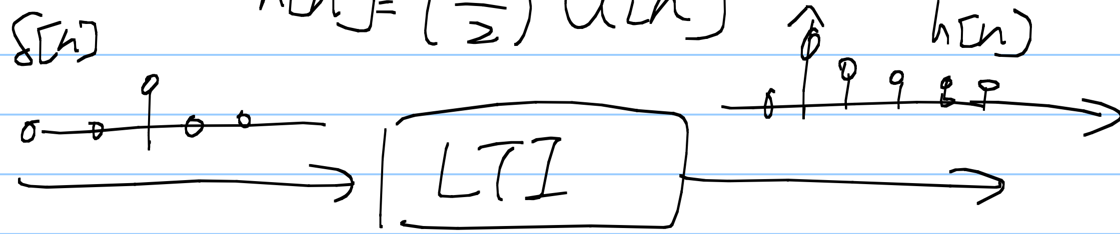
It is termed the convolution sum of $x[n]$ and $h[n]$, and is denoted by $y[n] = x[n] * h[n]$.

* **Theorem:** For a DT-LTI sys. with impulse response $h[n]$. The input/out relationship can be characterized by

$$y[n] = x[n] * h[n]$$
$$\triangleq \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Example: Suppose

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$



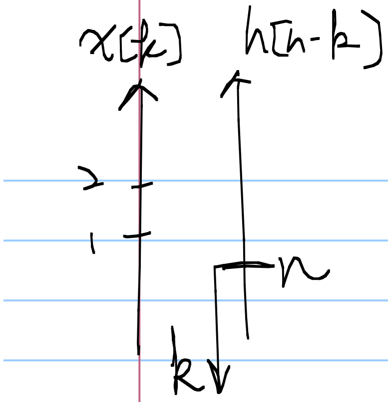
For a new input $x[n] = \begin{cases} n & \text{if } 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$

Find the output $y[n]$, and plot $y[n]$ vs. n.

Ans: $y[n] = x[n] * h[n]$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$ summing over k

$$h[n-k] = \left(\frac{1}{2}\right)^{n-k} u[n-k]$$
$$= \begin{cases} \left(\frac{1}{2}\right)^{n-k} & \text{if } n-k \ge 0 \iff k \le n \\ 0 & \text{if } n-k < 0 \iff k > n \end{cases}$$



Case 1: $n < 1$

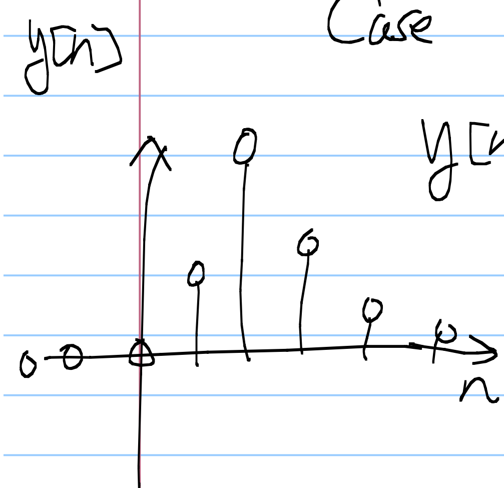
$y[n] = 0.$

Case 2: $n = 1$

$$y[1] = \sum_{k=1}^{1} x[k]\,h[n-k]$$

$$= 1 \times \left(\frac{1}{2}\right)^0 \times 1 = 1$$

Case 3: $2 \le n$

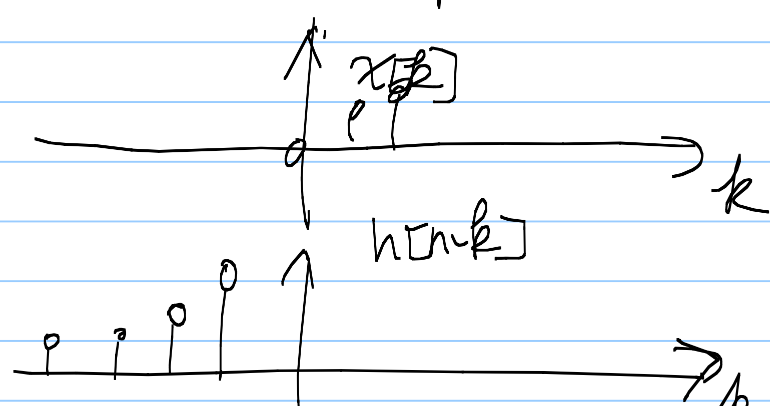


$$y[n] = \sum_{k=1}^{2} x[k]\,h[n-k]$$

$$= 1 \times \left(\frac{1}{2}\right)^{n-1} + 2 \times \left(\frac{1}{2}\right)^{n-2}$$

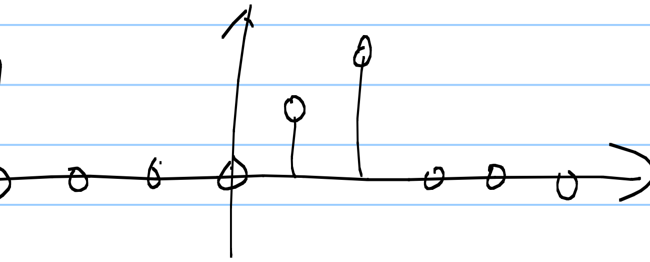
$$= 5 \times \left(\frac{1}{2}\right)^{n-1}$$

See Examples 2.2 to 2.4 for visualization of the above computation.

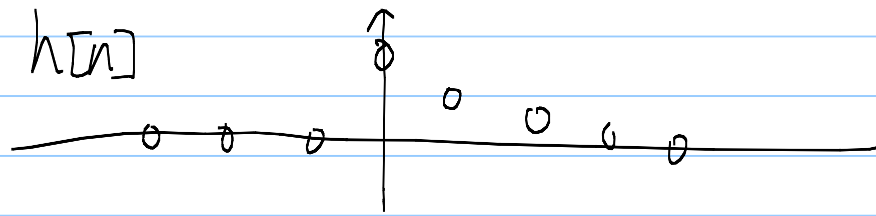


An alternative solution (conceptually simpler but computationally harder)

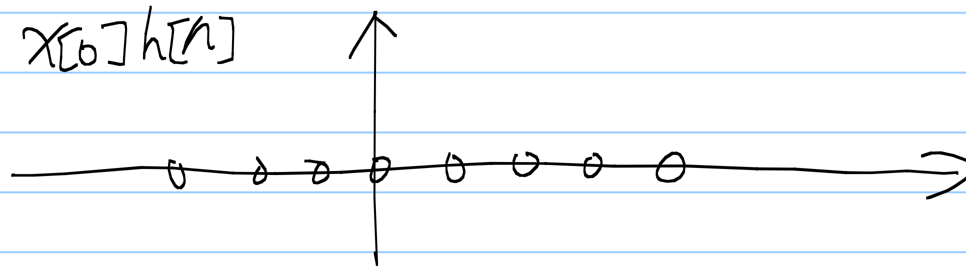
$x[n]$



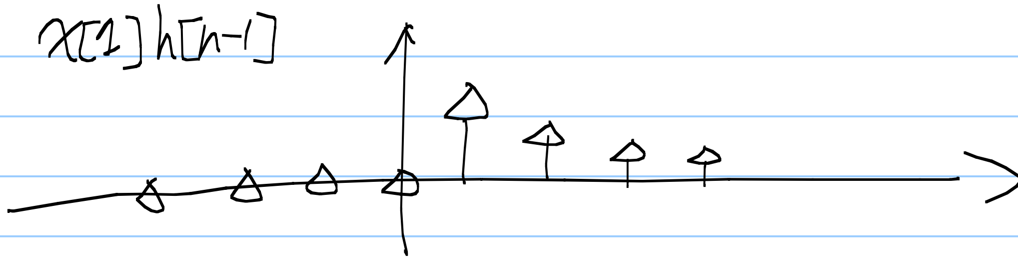
$h[n]$



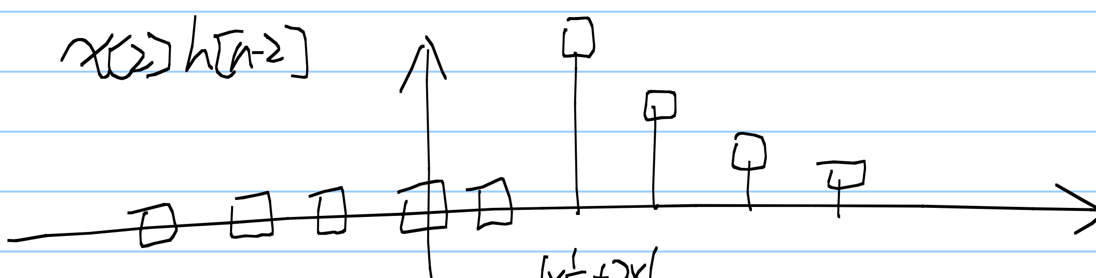
$x[0]\,h[n]$



$x[1]\,h[n-1]$



$x[2]\,h[n-2]$



Add them together