ECE 301-001 and 301-003, Midterm #3
8–9:30pm, Thursday, April 6, 2023, FRNY G140 and RHPH 172.

1. **Do not write answers on the back of pages!**
2. **After the exam ends, you will have 5 additional minutes to write down your name and Purdue ID on each of the pages.**
3. **If you need additional sheets of paper to write down your answers, please let one of the proctors know. We will hand out additional answer sheets as needed.**
4. Write your student ID number and signature in the space provided on this page.
5. This is a closed book exam. Neither calculators nor help sheets are allowed. A separate formula packet has been provided to you.
6. You have **90 minutes** to complete the exam. There are 6 multi-part questions.
7. You must **show all work** used to arrive at your answer. This is required to receive full credit, and also is helpful for you in getting partial credit.

Name:

Student ID:

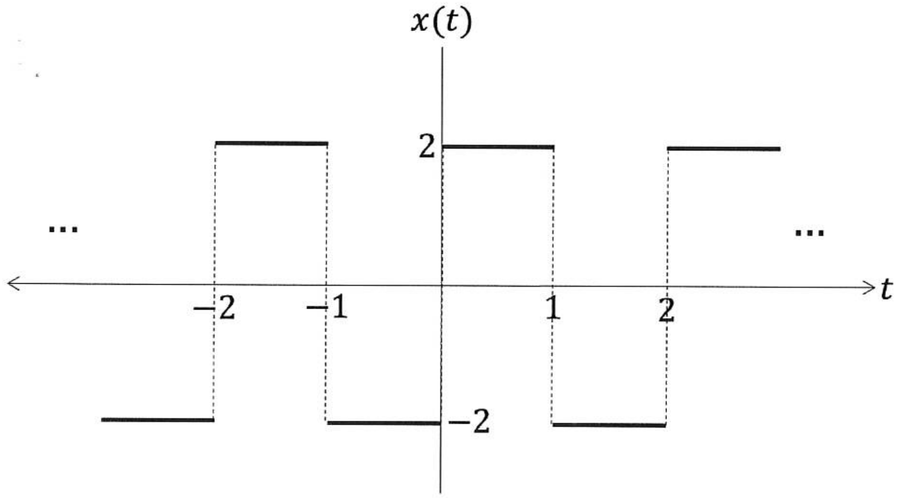
As a Boilermaker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date: 4/6/2023

Last Name: First Name: Purdue ID:

Question 1: [18%]

Consider the following periodic waveform $x(t)$:



(a) [10%] Let $\{a_k\}$ denote the Fourier series coefficients of $x(t)$. Determine the coefficients and write out the Fourier series expansion of $x(t)$.

(b) [8%] Plot the magnitude $|a_k|$ of the coefficients, for $-2 \leq k \leq 2$. Be sure to clearly label your axes.

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This sheet is for Question 1.

(a)

$T = 2$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\frac{2\pi}{T}t} dt = \frac{1}{2}\int_0^2 x(t) e^{-jk\pi t} dt$$

For $k = 0$,

$$a_0 = \frac{1}{2}\int_0^2 x(t) dt = \frac{1}{2}\int_0^1 2 dt + \frac{1}{2}\int_1^2 (-2) dt = 0$$

For $k \neq 0$,

$$a_k = \frac{1}{2}\int_0^2 x(t) e^{-jk\pi t} dt = \frac{1}{2}\int_0^1 2 e^{-jk\pi t} dt + \frac{1}{2}\int_1^2 (-2) e^{-jk\pi t} dt$$

$$= \left[-\frac{1}{jk\pi} e^{-jk\pi t}\right]_0^1 + \left[\frac{1}{jk\pi} e^{-jk\pi t}\right]_1^2$$

$$= \frac{1}{jk\pi}\left(1 - e^{-jk\pi} + e^{-j2k\pi} - e^{-jk\pi}\right)$$

$$= \frac{1}{jk\pi}\left(2 - 2\cdot(-1)^k\right) = \begin{cases} \frac{4}{jk\pi} & \text{when } k \text{ is odd} \\ 0 & \text{when } k \text{ is even} \end{cases}$$

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$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\pi t} \quad \text{where} \quad a_k = \begin{cases} \frac{4}{jk\pi} & \text{when } k \text{ is odd} \\ 0 & \text{when } k \text{ is even.} \end{cases}}$$

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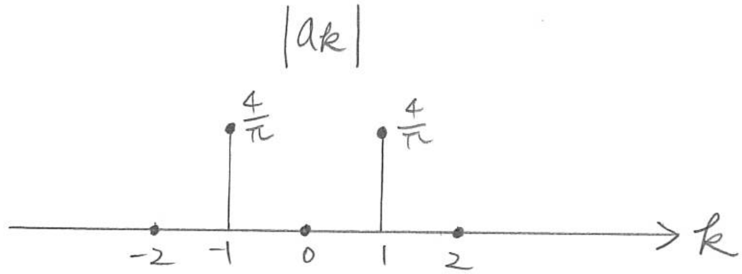
This sheet is for Question 1.

(b)

$$|a_k| = \begin{cases} \left|\frac{4}{jk\pi}\right| & \text{when } k \text{ is odd} \\ 0 & \text{when } k \text{ is even} \end{cases} = \begin{cases} \frac{4}{\pi}\frac{1}{|k|} & \text{when } k \text{ is odd} \\ 0 & \text{when } k \text{ is even} \end{cases}$$



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Question 2: [20%]

Consider a periodic signal $x(t)$ defined as follows:

$$x(t) = \begin{cases} 2t+2 & \text{if } |t| \le 1 \\ 0 & \text{if } 1 < |t| \le 8 \\ \text{periodic with period 16} & \end{cases}$$

This signal has $\omega_0 = \frac{\pi}{8}$ and Fourier series coefficients

$$a_0 = 0.25, \qquad a_k = j\frac{2\cos(k\pi/8)}{k\pi} + \frac{2\sin(k\pi/8)}{k\pi}\left(1 - \frac{8j}{k\pi}\right) \quad k \ne 0$$

(a) [3%] Plot $x(t)$ for the range of $-20 < t < 20$.

(b) [7%] Find the value of $\int_{t=0}^{16} x(t)e^{j0.5\pi t}dt$.

Now, consider a different signal $y(t)$:

$$y(t) = \begin{cases} 2t & \text{if } 0 \le t < 2 \\ 2t+4 & \text{if } -2 \le t < 0 \\ 0 & \text{if } 2 \le |t| \le 8 \\ \text{periodic with period 16} & \end{cases}$$

Denote its Fourier series coefficients by b_k.

(c) [4%] Plot $y(t)$ for the range of $-20 < t < 20$.

(d) [6%] Find the value of b_{-11}.

[Hint 1: Your answer for b_{-11} should be an explicit value, say something like $b_{-11} = j\sin(12.7\pi)\cos^2(-\sqrt{\pi})/\sqrt{2}$.]

[Hint 2: If you do not know how to find the value of b_{-11}, you can state the general relationship between b_k and a_k. You will receive 4.5 points if your answer is correct.]

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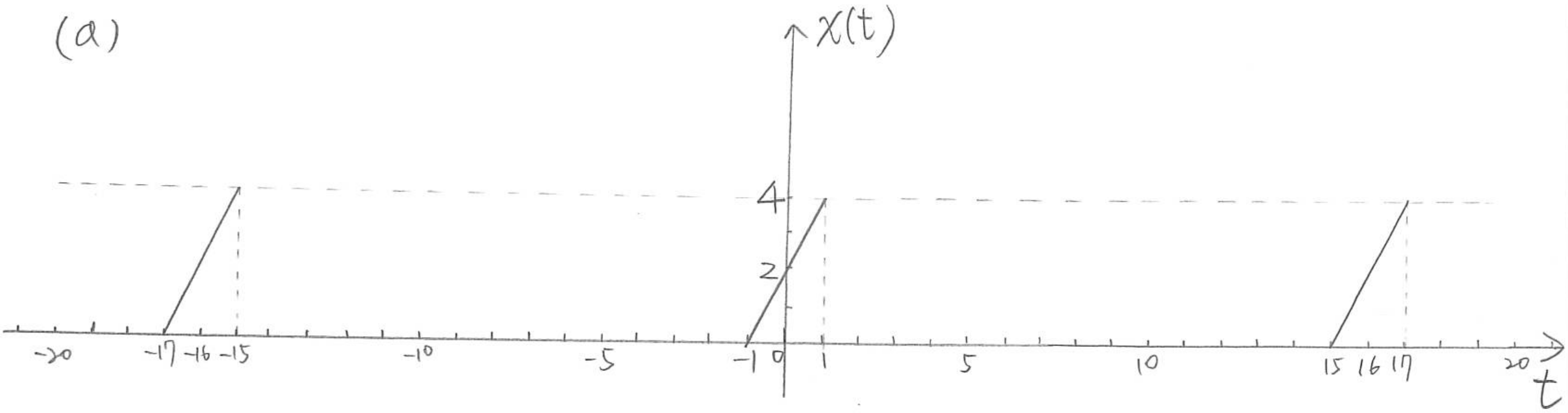
This sheet is for Question 2.

(a)



(b) $\int_{t=0}^{16} x(t) e^{j0.5\pi t} dt$

$= 16 \cdot \frac{1}{16} \int_{t=0}^{16} x(t) e^{-j(-4)\frac{\pi}{8}t} dt$

$= 16 \cdot a_{-4}$

$= 16 \cdot \left[j \frac{2\cos(\frac{-4}{8}\pi)}{-4\pi} + \frac{2\sin(\frac{-4}{8}\pi)}{-4\pi} \left(1 - \frac{8j}{-4\pi}\right) \right]$

$= 16 \left[0 + \frac{1}{2\pi} \left(1 + \frac{2}{\pi} j\right) \right]$

$= \boxed{\frac{8}{\pi} \left(1 + \frac{2}{\pi} j\right)}$ #

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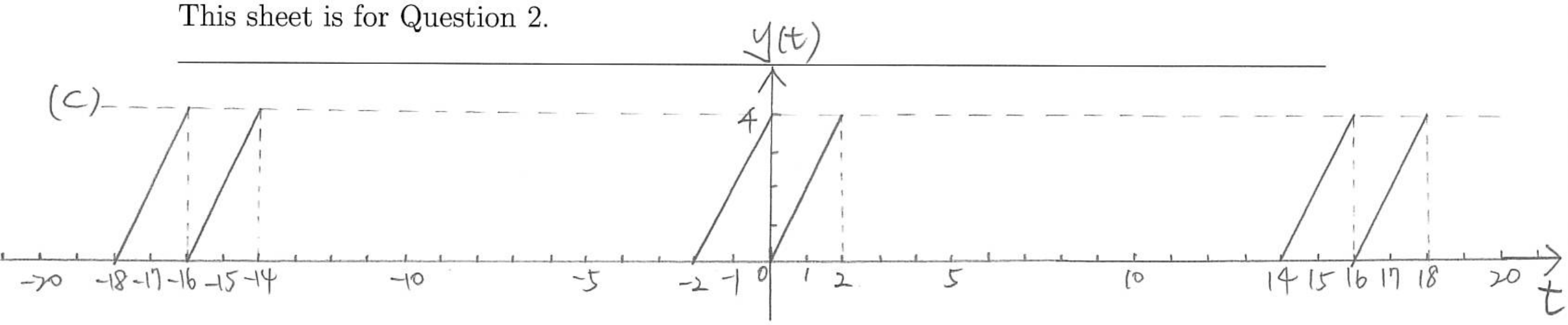
This sheet is for Question 2.

(c)



(d) $y(t) = x(t+1) + x(t-1)$

Apply time shifting property:

$$b_k = a_k e^{jk\frac{2\pi}{16}\cdot 1} + a_k e^{-jk\frac{2\pi}{16}\cdot 1}$$

$$= a_k e^{jk\frac{\pi}{8}} + a_k e^{-jk\frac{\pi}{8}}$$

$$= a_k \cdot 2\cos\left(k\frac{\pi}{8}\right)$$

$$b_{-11} = a_{-11} \cdot 2\cos\left(-11\cdot\frac{\pi}{8}\right)$$

$$= \left[j\frac{2\cos(-\frac{11}{8}\pi)}{-11\pi} + \frac{2\sin(-\frac{11}{8}\pi)}{-11\pi}\left(1+j\frac{8}{11\pi}\right)\right]\cdot 2\cos\left(-\frac{11}{8}\pi\right)$$

$$= \left[-j\frac{2\cos(\frac{11}{8}\pi)}{11\pi} + \frac{2\sin(\frac{11}{8}\pi)}{11\pi}\left(1+j\frac{8}{11\pi}\right)\right]\cdot 2\cos\left(\frac{11}{8}\pi\right)$$

$$= \boxed{-j\frac{4\cos^2(\frac{11}{8}\pi)}{11\pi} + \frac{2\sin(\frac{11}{4}\pi)}{11\pi}\left(1+j\frac{8}{11\pi}\right)}$$

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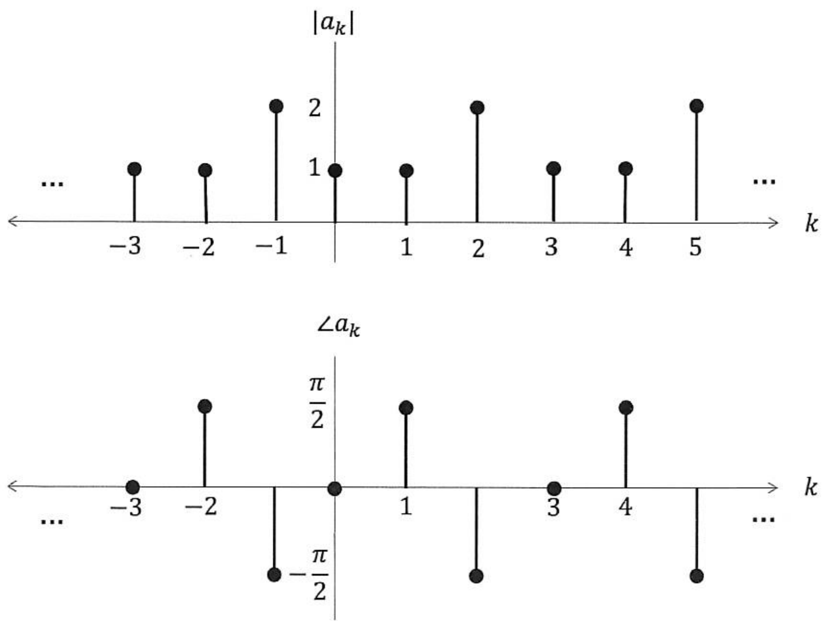
Question 3: [17%]

Consider a periodic discrete-time signal $x[n]$ which has the following Fourier series coefficients $\{a_k\}$:



Notice that the DTFS sequence $\{a_k\}$ is periodic with a fundamental period of 3.

(a) [3%] What is the fundamental period of $x[n]$? Explain in 1 sentence.

(b) [6%] Write out the Fourier series expansion of $x[n]$.

(c) [5%] Let $y[n] = x[-n]$, and let $\{b_k\}$ be the Fourier series coefficients of $y[n]$. Plot $|b_k|$ for $-3 \le k \le 5$.

(d) [3%] Express the Fourier series of $x[n]$ from (b) in the simplified form

$$x[n] = A + B\sin(k_B\omega_0 n + \phi_B) + Ce^{j(k_C\omega_0 n + \phi_C)}$$

where ω_0 is the fundamental frequency and $A, B, C, k_B, \phi_B, k_C, \phi_C$ are values that you have determined.

[Hint: Consider coming back to do (d) after you finish the rest of the exam, because it is only 3% and may take some time.]

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This sheet is for Question 3.

(a) The fundamental period of $x[n]$ is $N=3$

$\because$ Given $a_0 = a_3$, we must have $\frac{1}{N}\sum_{n=\langle N\rangle} x[n] = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-j3(\frac{2\pi}{N})n}$

$\therefore\ e^{-j3(\frac{2\pi}{N})} = 1 \Rightarrow N = 3$

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(b)

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(\frac{2\pi}{N})n} = \sum_{k=0}^{2} a_k e^{jk\frac{2\pi}{3}n} = \sum_{k=-1}^{1} a_k e^{jk\frac{2\pi}{3}n}$$

$$= 1 + e^{j\frac{-\pi}{2}} e^{j\frac{-2\pi}{3}n} + 2e^{-j\frac{\pi}{2}} e^{-j\frac{2\pi}{3}n}$$

$$= 1 + 2\cos\left(\frac{2\pi}{3}n + \frac{\pi}{2}\right) + e^{-j\left(\frac{2\pi}{3}n + \frac{\pi}{2}\right)}$$

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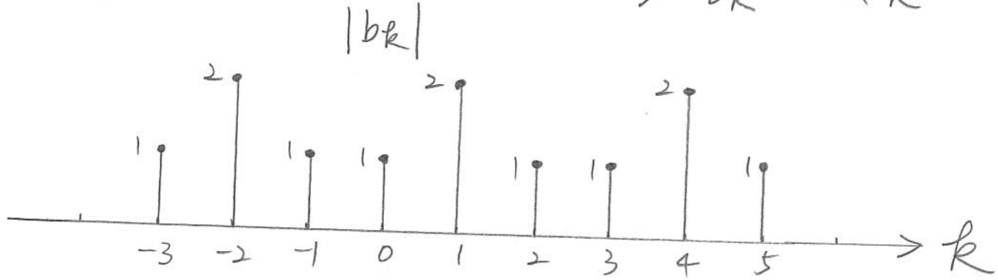
(c) $y[n] = x[-n] \xrightarrow{\text{by Time Reversal Property}} b_k = a_{-k}$



(d) Note that $\cos(\theta + \frac{\pi}{2}) = -\sin(\theta)$

From (b), we have

$$x[n] = 1 - 2\sin\left(\frac{2\pi}{3}n + 0\right) + e^{j\left(-\frac{2\pi}{3}n - \frac{\pi}{2}\right)}$$

$$= 1 + 2\sin\left(-\frac{2\pi}{3}n + 0\right) + 1\, e^{j\left(-\frac{2\pi}{3}n - \frac{\pi}{2}\right)}$$

$\because\ \omega_0 = \frac{2\pi}{3}$

$\therefore\ A = 1,\ B = 2,\ C = 1,\ k_B = -1,\ \phi_B = 0$

$k_C = -1,\ \phi_C = -\frac{\pi}{2}$

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This sheet is for Question 3.

Last Name: First Name: Purdue ID:

Question 4: [14%]

Consider a DT LTI system with impulse response

$$h[n] = \begin{cases} 2.5 & \text{if } 0 \le n \le 19 \\ 0 & \text{otherwise} \end{cases}$$

(a) [7%] Compute the following summation: $\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$
Hint: The geometric series formula may be useful: if $r \neq 1$, then we have

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}$$

Also consider a DT periodic signal $x[n]$ with period 80 and

$$x[n] = \begin{cases} n & 1 \le n \le 80 \\ \text{periodic with period 80} & \end{cases}$$

Suppose we use $x[n]$ as the input to the above system. Denote the output by $y[n]$.

(b) [7%] Denote the DTFS coefficients of $y[n]$ by $\{b_k\}$. Show that $b_8 = 0$.

[Hint 1: You don't need to find the DTFS coefficients of $x[n]$ or the general expression for b_k to solve this question. All you need to show is that $b_8 = 0$.]

[Hint 2: If you don't know how to answer to this question, please find the output of the above LTI system, denoted by $y_2[n]$, when the input is $x_2[n] = e^{j\frac{2\pi}{5}n}$. You will receive 5.5 points if your answer is correct.]

Last Name: First Name: Purdue ID:
This sheet is for Question 4.

(a)
$$\sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = \sum_{n=0}^{19} \frac{5}{2} e^{-j\omega n} = \sum_{k=1}^{20} \frac{5}{2} (e^{-j\omega})^{k-1}$$

$$= \frac{\frac{5}{2}(1-(e^{-j\omega})^{20})}{1-e^{-j\omega}} = \boxed{\frac{5}{2} \frac{1-e^{-j20\omega}}{1-e^{-j\omega}}} \; \#$$

(b) Denote the Fourier coefficients of $x[n]$ as a_k
and the discrete-time Fourier transform of $h[n]$ as $H(e^{j\omega})$

$$b_8 = a_8 H(e^{j8\omega_0}) \text{ where } \omega_0 = \frac{2\pi}{80} = \frac{\pi}{40}$$

$$H(e^{j8\frac{\pi}{40}}) = H(e^{j\frac{\pi}{5}}) = \frac{5}{2} \frac{1-e^{-j20\cdot\frac{\pi}{5}}}{1-e^{-j\frac{\pi}{5}}}$$

$$= \frac{5}{2} \frac{1-1}{1-e^{-j\frac{\pi}{5}}} = 0$$

$$\therefore \; b_8 = a_8 H(e^{j\frac{\pi}{5}}) = 0 \qquad \text{Q.E.D.}$$

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This sheet is for Question 4.

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Question 5: [12%]

Consider a CT aperiodic signal

$$x(t) = \begin{cases} 12 & \text{if } -3 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) [7%] Denote the CTFT of $x(t)$ by $X(j\omega)$. Plot $X(j\omega)$ for the range of $-\pi \le \omega \le \pi$. Please carefully mark the intersecting points to the horizontal and vertical axes.

Consider a second CT aperiodic signal

$$y(t) = \begin{cases} 12e^{j\frac{2\pi}{3}t} & \text{if } -3 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$$

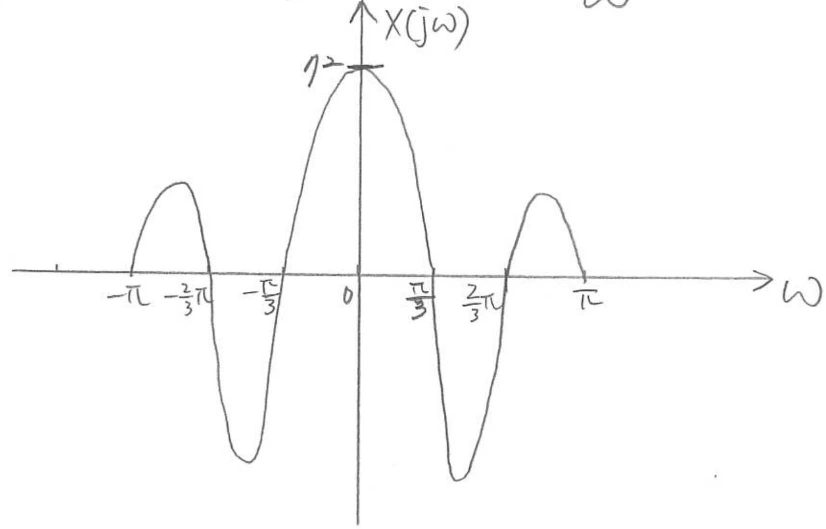
(b) [5%] Denote the CTFT of $y(t)$ by $Y(j\omega)$. Plot $Y(j\omega)$ for the range of $-\pi \le \omega \le \pi$.

[Hint: If you do not know how to draw the plot in (b), you can write down the relationship between $X(j\omega)$ and $Y(j\omega)$ and you will receive 4 points if your answer is correct.]

(a)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-3}^{3} 12 e^{-j\omega t} dt = 12 \left[\frac{1}{-j\omega} e^{-j\omega t} \right]_{t=-3}^{3}$$

$$= -\frac{12}{j\omega} \left(e^{-j3\omega} - e^{j3\omega} \right) = 24 \frac{\sin 3\omega}{\omega}$$



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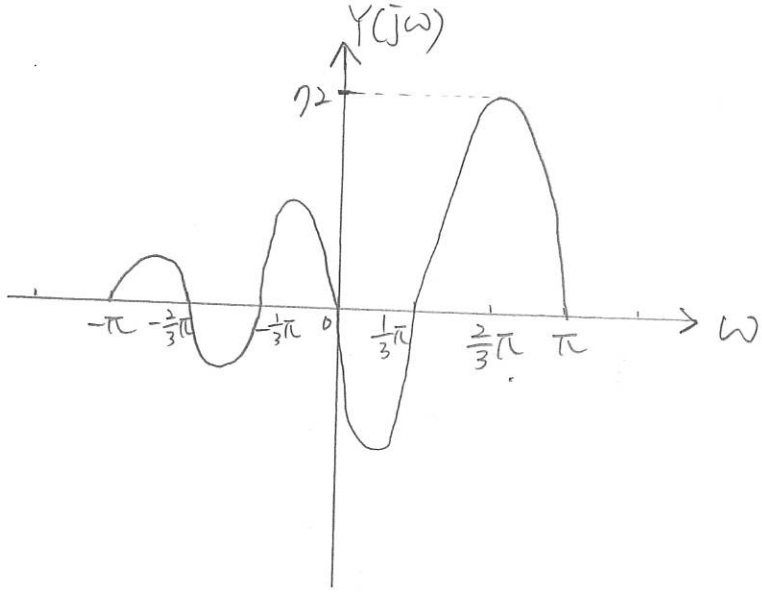
This sheet is for Question 5.

(b) $y(t)$ is formed by a frequency shifting of $x(t)$ with $\omega_0 = \frac{2\pi}{3}$

$$\Rightarrow Y(j\omega) = X\left(j\left(\omega - \frac{2\pi}{3}\right)\right)$$



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This sheet is for Question 5.

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Question 6: [19%]

Consider a CT LTI system with frequency response

$$H(j\omega) = \frac{1}{2 + j\omega}$$

(a) [4%] Is the system invertible? Justify your answer.

(b) [5%] Determine the impulse response $h(t)$ of the system.

[Hint: In this part and the rest of Q6, you are strongly encouraged to use the Fourier transform tables.]

Now, suppose we send an input

$$x(t) = e^{-3t}u(t)$$

through the system to get an output $y(t)$.

(c) [5%] Determine the frequency spectrum $Y(j\omega)$ of the output.

Finally, suppose we pass another input $x_2(t)$ to the system and measure a frequency spectrum at the output of

$$Y_2(j\omega) = 3 - \delta(\omega - 5\pi) - \delta(\omega + 5\pi) + \frac{9}{(6 + 3j\omega)^2}$$

(d) [5%] Find an expression for the output $y_2(t)$.

(a) Yes, $\because H(j\omega) \neq 0$ for all ω.

(b) By Table 4.2, $\mathcal{F}\{e^{-at}u(t)\} = \frac{1}{a+j\omega}$

$\therefore \boxed{h(t) = e^{-2t}u(t)}$ #

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This sheet is for Question 6.

(c) $x(t) = e^{-3t}u(t) \longleftrightarrow X(j\omega) = \frac{1}{3+j\omega}$

$y(t) = x(t) * h(t) \longleftrightarrow Y(j\omega) = X(j\omega)H(j\omega)$

$$= \boxed{\frac{1}{(3+j\omega)(2+j\omega)}}$$

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(d) $Y_2(j\omega) = 3 - \delta(\omega - 5\pi) - \delta(\omega + 5\pi) + \frac{9}{9(2+j\omega)^2}$

$$= 3 - \delta(\omega - 5\pi) - \delta(\omega + 5\pi) + \frac{1}{(2+j\omega)^2}$$

By Table 4.2,

$$y_2(t) = 3\delta(t) - \frac{1}{2\pi}e^{j5\pi t} - \frac{1}{2\pi}e^{-j5\pi t} + te^{-2t}u(t)$$

$$= 3\delta(t) - \frac{1}{2\pi} \cdot 2\cos(5\pi t) + te^{-2t}u(t)$$

$$= \boxed{3\delta(t) - \frac{1}{\pi}\cos(5\pi t) + te^{-2t}u(t)}$$

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Last Name: First Name: Purdue ID:
This sheet is for Question 6.
