ECE 301-001 and 301-003, Midterm #3
8–9:30pm, Thursday, April 6, 2023, FRNY G140 and RHPH 172.

1. **Do not write answers on the back of pages!**
2. **After the exam ends, you will have 5 additional minutes to write down your name and Purdue ID on each of the pages.**
3. **If you need additional sheets of paper to write down your answers, please let one of the proctors know. We will hand out additional answer sheets as needed.**
4. Write your student ID number and signature in the space provided on this page.
5. This is a closed book exam. Neither calculators nor help sheets are allowed. A separate formula packet has been provided to you.
6. You have **90 minutes** to complete the exam. There are 6 multi-part questions.
7. You must **show all work** used to arrive at your answer. This is required to receive full credit, and also is helpful for you in getting partial credit.

Name:

Student ID:

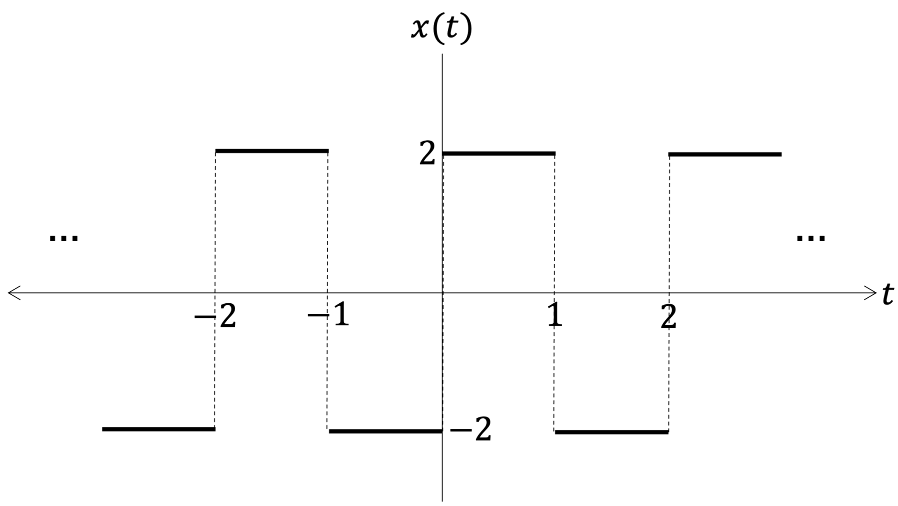
As a Boilermaker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Last Name: First Name: Purdue ID:

Question 1: [18%]

Consider the following periodic waveform $x(t)$:



(a) [10%] Let $\{a_k\}$ denote the Fourier series coefficients of $x(t)$. Determine the coefficients and write out the Fourier series expansion of $x(t)$.

(b) [8%] Plot the magnitude $|a_k|$ of the coefficients, for $-2 \leq k \leq 2$. Be sure to clearly label your axes.

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Question 2: [20%]

Consider a periodic signal $x(t)$ defined as follows:

$$x(t) = \begin{cases} 2t+2 & \text{if } |t| \leq 1 \\ 0 & \text{if } 1 < |t| \leq 8 \\ \text{periodic with period 16} & \end{cases}$$

This signal has $\omega_0 = \frac{\pi}{8}$ and Fourier series coefficients

$$a_0 = 0.25, \qquad a_k = j\frac{2\cos(k\pi/8)}{k\pi} + \frac{2\sin(k\pi/8)}{k\pi}\left(1 - \frac{8j}{k\pi}\right) \quad k \neq 0$$

(a) [3%] Plot $x(t)$ for the range of $-20 < t < 20$.

(b) [7%] Find the value of $\int_{t=0}^{16} x(t)e^{j0.5\pi t}dt$.

Now, consider a different signal $y(t)$:

$$y(t) = \begin{cases} 2t & \text{if } 0 \leq t < 2 \\ 2t+4 & \text{if } -2 \leq t < 0 \\ 0 & \text{if } 2 \leq |t| \leq 8 \\ \text{periodic with period 16} & \end{cases}$$

Denote its Fourier series coefficients by b_k.

(c) [4%] Plot $y(t)$ for the range of $-20 < t < 20$.

(d) [6%] Find the value of b_{-11}.

[Hint 1: Your answer for b_{-11} should be an explicit value, say something like $b_{-11} = j\sin(12.7\pi)\cos^2(-\sqrt{\pi})/\sqrt{2}$.]

[Hint 2: If you do not know how to find the value of b_{-11}, you can state the general relationship between b_k and a_k. You will receive 4.5 points if your answer is correct.]

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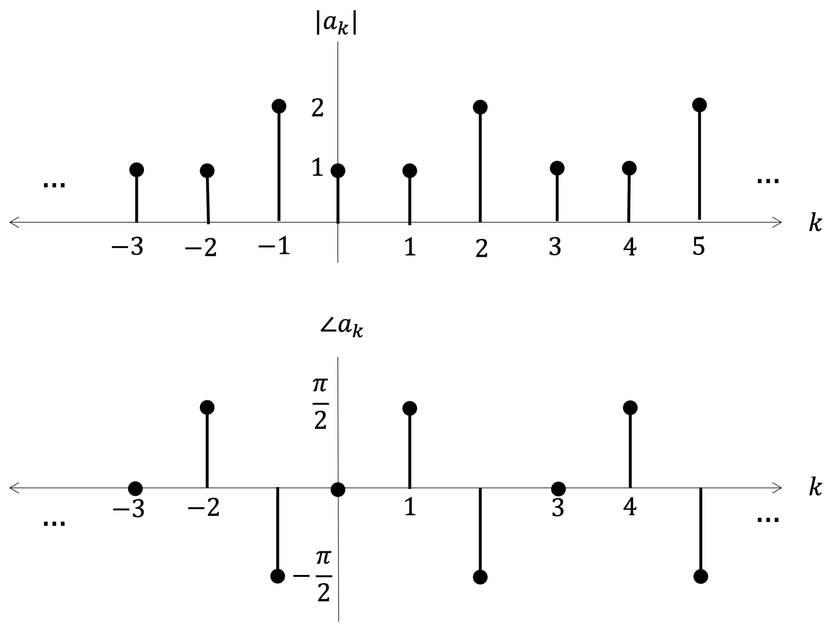
Question 3: [17%]

Consider a periodic discrete-time signal $x[n]$ which has the following Fourier series coefficients $\{a_k\}$:



Notice that the DTFS sequence $\{a_k\}$ is periodic with a fundamental period of 3.

(a) [3%] What is the fundamental period of $x[n]$? Explain in 1 sentence.

(b) [6%] Write out the Fourier series expansion of $x[n]$.

(c) [5%] Let $y[n] = x[-n]$, and let $\{b_k\}$ be the Fourier series coefficients of $y[n]$. Plot $|b_k|$ for $-3 \leq k \leq 5$.

(d) [3%] Express the Fourier series of $x[n]$ from (b) in the simplified form

$$x[n] = A + B\sin(k_B\omega_0 n + \phi_B) + Ce^{j(k_C\omega_0 n+\phi_C)}$$

where ω_0 is the fundamental frequency and $A, B, C, k_B, \phi_B, k_C, \phi_C$ are values that you have determined.

[Hint: Consider coming back to do (d) after you finish the rest of the exam, because it is only 3% and may take some time.]

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Question 4: [14%]

Consider a DT LTI system with impulse response

$$h[n] = \begin{cases} 2.5 & \text{if } 0 \leq n \leq 19 \\ 0 & \text{otherwise} \end{cases}$$

(a) [7%] Compute the following summation: $\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$

Hint: The geometric series formula may be useful: if $r \neq 1$, then we have

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}$$

Also consider a DT periodic signal $x[n]$ with period 80 and

$$x[n] = \begin{cases} n & 1 \leq n \leq 80 \\ \text{periodic with period 80} & \end{cases}$$

Suppose we use $x[n]$ as the input to the above system. Denote the output by $y[n]$.

(b) [7%] Denote the DTFS coefficients of $y[n]$ by $\{b_k\}$. Show that $b_8 = 0$.

[Hint 1: You don't need to find the DTFS coefficients of $x[n]$ or the general expression for b_k to solve this question. All you need to show is that $b_8 = 0$.]

[Hint 2: If you don't know how to answer to this question, please find the output of the above LTI system, denoted by $y_2[n]$, when the input is $x_2[n] = e^{j\frac{2\pi}{5}n}$. You will receive 5.5 points if your answer is correct.]

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Question 5: [12%]

Consider a CT aperiodic signal

$$x(t) = \begin{cases} 12 & \text{if } -3 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) [7%] Denote the CTFT of $x(t)$ by $X(j\omega)$. Plot $X(j\omega)$ for the range of $-\pi \leq \omega \leq \pi$. Please carefully mark the intersecting points to the horizontal and vertical axes.

Consider a second CT aperiodic signal

$$y(t) = \begin{cases} 12e^{j\frac{2\pi}{3}t} & \text{if } -3 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(b) [5%] Denote the CTFT of $y(t)$ by $Y(j\omega)$. Plot $Y(j\omega)$ for the range of $-\pi \leq \omega \leq \pi$.

[Hint: If you do not know how to draw the plot in (b), you can write down the relationship between $X(j\omega)$ and $Y(j\omega)$ and you will receive 4 points if your answer is correct.]

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Question 6: [19%]

Consider a CT LTI system with frequency response

$$H(j\omega) = \frac{1}{2 + j\omega}$$

(a) [4%] Is the system invertible? Justify your answer.

(b) [5%] Determine the impulse response $h(t)$ of the system.

[Hint: In this part and the rest of Q6, you are strongly encouraged to use the Fourier transform tables.]

Now, suppose we send an input

$$x(t) = e^{-3t}u(t)$$

through the system to get an output $y(t)$.

(c) [5%] Determine the frequency spectrum $Y(j\omega)$ of the output.

Finally, suppose we pass another input $x_2(t)$ to the system and measure a frequency spectrum at the output of

$$Y_2(j\omega) = 3 - \delta(\omega - 5\pi) - \delta(\omega + 5\pi) + \frac{9}{(6 + 3j\omega)^2}$$

(d) [5%] Find an expression for the output $y_2(t)$.

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Discrete-time Fourier series

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t}\, dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t}\, d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\left\{\begin{array}{l} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{array}\right.$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\left\{\begin{array}{ll} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{array}\right.$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & [\text{x[n] real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} & [\text{x[n] real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(j\omega)$
		$y(t)$	$Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{Re}\{X(j\omega)\} = \mathcal{Re}\{X(-j\omega)\} \\ \mathcal{Im}\{X(j\omega)\} = -\mathcal{Im}\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(j\omega)\}$ $j\mathcal{Im}\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—