ECE 301-001 and 301-003, Midterm #2
8–9:30pm, Wednesday, March 1, 2023, CL50 Rm224.

1. **Do not write answers on the back of pages!**
2. **After the exam ends, you will have 5 additional minutes to write down your name and Purdue ID on each of the pages.**
3. **If you need additional sheets of paper to write down your answers, please let one of the proctors know. We will hand out additional answer sheets as needed.**
4. Write your student ID number and signature in the space provided on this page.
5. This is a closed book exam. Neither calculators nor help sheets are allowed.
6. You have **90 minutes** to complete the exam. There are 6 multi-part questions.
7. You must **show all work** used to arrive at your answer. This is required to receive full credit, and also is helpful for you in getting partial credit.

Name:

Student ID:

As a Boilermaker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date: 3/1/2023

Last Name: First Name: Purdue ID:

Question 1: [18%] Consider an LTI system with impulse response

$$h(t) = \begin{cases} 0.5^{(t-4)} & \text{if } t \geq 1 \\ 0 & \text{if } t < 1 \end{cases}.$$

(a) [13%] If the input is

$$x_1(t) = u(-t+1),$$

where $u(t)$ is the unit step signal, denote the corresponding output by $y_1(t)$. Find the expression of $y_1(t)$.

[Hint: the following equality may be useful: $0.5 = e^{-\ln(2)}$.]

(b) [5%] Now, suppose we change the input to

$$x_2(t) = \begin{cases} 0 & \text{if } t < -1 \\ 2 & \text{if } -1 \leq t < 2 \\ 3 & \text{if } t \geq 2 \end{cases}.$$

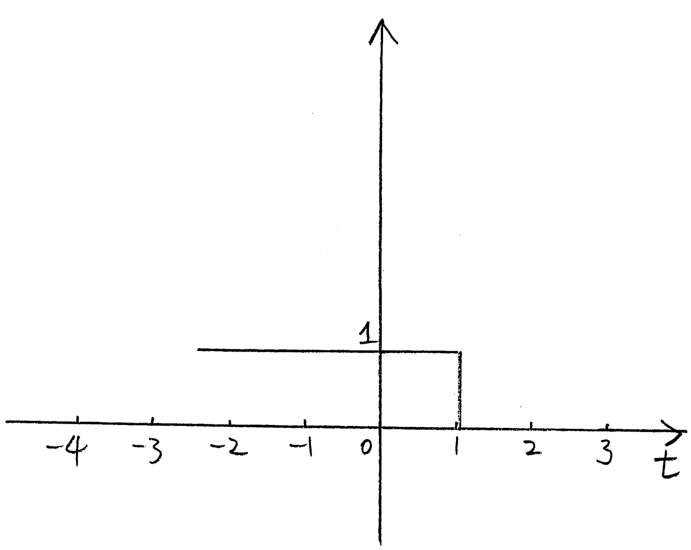
Denote the corresponding output by $y_2(t)$. Write down the expression of $y_2(t)$.

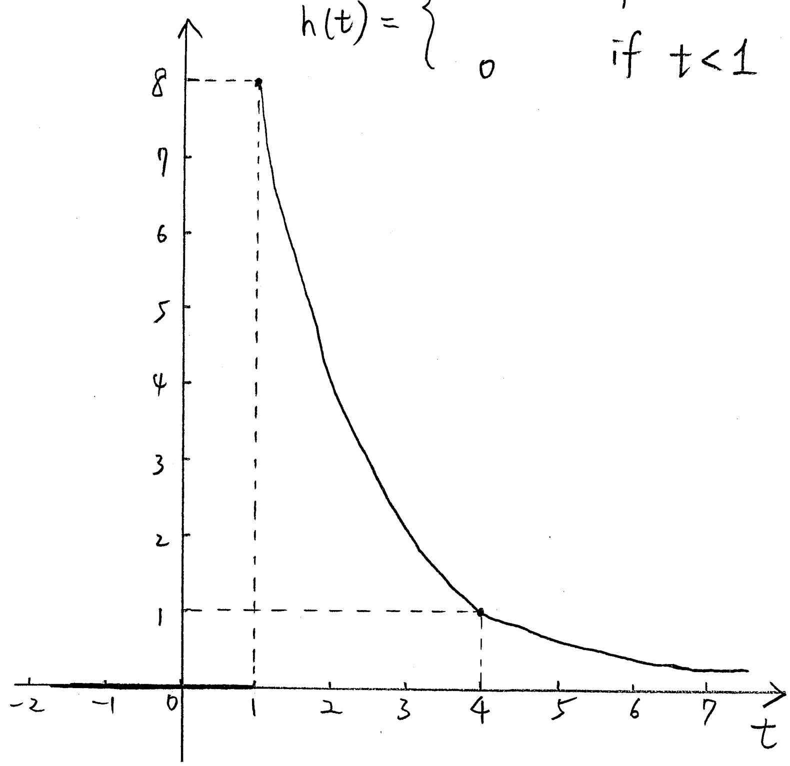
[Hint: For part (b), rather than doing a direct computation, consider the relationship between $y_2(t)$ and $y_1(t)$. You will receive 4% just for correctly identifying this relationship even if you are unable to find the expression of $y_2(t)$.]

$$h(t) = \begin{cases} \frac{1}{2}^{(t-4)} & \text{if } t \geq 1 \\ 0 & \text{if } t < 1 \end{cases}$$

(a)

$x_1(t) = u(-t+1)$





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This sheet is for Question 1.

$$y_1(t) = x_1(t) * h(t) = \int_{-\infty}^{\infty} x_1(t-\tau)h(\tau)d\tau$$

$$= \int_1^{\infty} u(-t+\tau+1)\left(\tfrac{1}{2}\right)^{(\tau-4)} d\tau$$

$$= \begin{cases} \int_1^{\infty} \left(\tfrac{1}{2}\right)^{(\tau-4)} d\tau & \text{if } t-1 < 1 \\ \int_{t-1}^{\infty} \left(\tfrac{1}{2}\right)^{(\tau-4)} d\tau & \text{if } t-1 \geq 1 \end{cases}$$

$$\int_1^{\infty} \left(\tfrac{1}{2}\right)^{(\tau-4)} d\tau = \int_1^{\infty} e^{-\ln(2)(\tau-4)} d\tau = -\frac{1}{\ln(2)}\left[e^{-\ln(2)(\tau-4)}\right]_1^{\infty}$$

$$= \frac{1}{\ln(2)} e^{3\ln(2)} = \frac{8}{\ln(2)}$$

$$\int_{t-1}^{\infty} \left(\tfrac{1}{2}\right)^{(\tau-4)} d\tau = \int_{t-1}^{\infty} e^{-\ln(2)(\tau-4)} d\tau = -\frac{1}{\ln(2)}\left[e^{-\ln(2)(\tau-4)}\right]_{t-1}^{\infty}$$

$$= \frac{1}{\ln(2)} e^{-\ln(2)(t-5)} = \frac{2^{(5-t)}}{\ln(2)}$$

Hence,

$$\boxed{\begin{aligned} y_1(t) &= \begin{cases} \frac{8}{\ln(2)} & \text{if } t < 2 \\ \frac{2^{(5-t)}}{\ln(2)} & \text{if } t \geq 2 \end{cases} \\ &= \frac{8}{\ln(2)} u(-t+2) + \frac{2^{(5-t)}}{\ln(2)} u(t-2) \end{aligned}}$$

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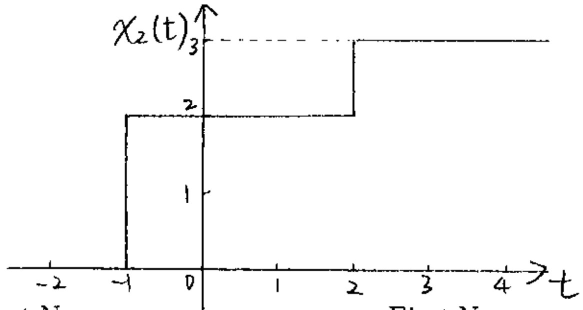

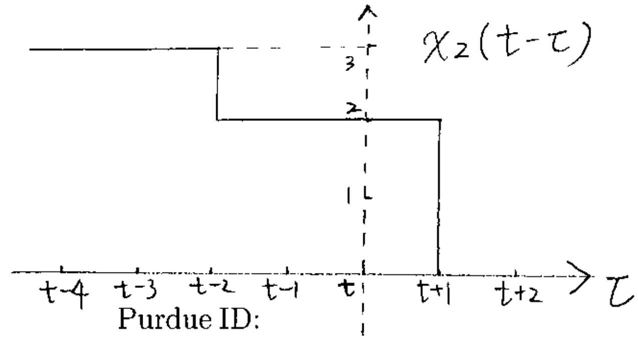

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This sheet is for Question 1.

$$y_2(t) = x_2(t) * h(t) = \int_{-\infty}^{\infty} x_2(t-\tau)h(\tau)\,d\tau$$

If $t+1 < 1 \Rightarrow t < 0$, we have $y_2(t) = 0$

If $t+1 \geq 1$ and $t-2 < 1 \Rightarrow 0 \leq t < 3$

$$y_2(t) = \int_1^{t+1} 2\cdot\left(\tfrac{1}{2}\right)^{(\tau-4)} d\tau = 2^5 \cdot \int_1^{t+1} 2^{-\tau} d\tau = 2^5 \cdot \int_1^{t+1} e^{-\ln(2)\cdot\tau} d\tau$$

$$= -\frac{2^5}{\ln(2)}\left[e^{-\ln(2)\cdot\tau}\right]_{\tau=1}^{t+1} = \frac{2^5}{\ln(2)}\left(e^{-\ln(2)\cdot 1} - e^{-\ln(2)(t+1)}\right)$$

$$= \frac{2^5}{\ln(2)}\left(2^{-1} - 2^{-t-1}\right) = \frac{16\,(1-2^{-t})}{\ln(2)}$$

If $t-2 \geq 1 \Rightarrow t \geq 3$

$$y_2(t) = \int_1^{t-2} 3\cdot\left(\tfrac{1}{2}\right)^{(\tau-4)} d\tau + \int_{t-2}^{t+1} 2\cdot\left(\tfrac{1}{2}\right)^{(\tau-4)} d\tau$$

$$= \frac{3\cdot 2^4}{\ln(2)}\left(e^{-\ln(2)\cdot 1} - e^{-\ln(2)(t-2)}\right) + \frac{2^5}{\ln(2)}\left(e^{-\ln(2)(t-2)} - e^{-\ln(2)(t+1)}\right)$$

$$= \frac{3\cdot 2^4}{\ln(2)}\left(2^{-1} - 2^{(2-t)}\right) + \frac{2^5}{\ln(2)}\left(2^{(2-t)} - 2^{(-t-1)}\right)$$

$$= \frac{8\cdot(3-10\cdot 2^{-t})}{\ln(2)}$$

Hence,

$$\boxed{y_2(t) = \begin{cases} 0 & \text{if } t < 0 \\ \dfrac{16\,(1-2^{-t})}{\ln(2)} & \text{if } 0 \leq t < 3 \\ \dfrac{8(3-10\cdot 2^{-t})}{\ln(2)} & \text{if } t \geq 3 \end{cases}}$$

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Last Name: First Name: Purdue ID:

Question 2: [16%] Consider an LTI system with impulse response

$$h(t) = \begin{cases} 0.5e^{-t} & \text{if } 0 \le t \le 1 \\ 0.5e^{t} & \text{if } -1 \le t < 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) [8%] Find the frequency response $H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$ of the system.

(b) [8%] If we input the signal

$$x(t) = \frac{e^{j5t} + e^{-j5t}}{2} + 3e^{j2\pi t}$$

to the system, find the output $y(t)$.

[Hint: Your answers can be something like $\frac{e^{(1+j0.5)t}}{2-\pi j} - 5je^{-3t}$. There is no need to further simplify them.]

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This sheet is for Question 2.

(a)

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-1}^{0} \frac{1}{2} e^{\tau} e^{-j\omega\tau} d\tau + \int_{0}^{1} \frac{1}{2} e^{-\tau} e^{-j\omega\tau} d\tau$$

$$= \frac{1}{2} \int_{-1}^{0} e^{(1-j\omega)\tau} d\tau + \frac{1}{2} \int_{0}^{1} e^{(-1-j\omega)\tau} d\tau$$

$$= \frac{1}{2} \left\{ \frac{1}{(1-j\omega)} \left[e^{(1-j\omega)\tau} \right]_{-1}^{0} + \frac{1}{-1-j\omega} \left[e^{(-1-j\omega)\tau} \right]_{0}^{1} \right\}$$

$$= \frac{1}{2} \left\{ \frac{1}{1-j\omega} (1 - e^{-1+j\omega}) + \frac{1}{1+j\omega} \cdot (1 - e^{-1-j\omega}) \right\}$$

$$= \frac{1}{2} \left\{ \frac{1+j\omega}{1+\omega^2} - \frac{1+j\omega}{1+\omega^2} e^{-1+j\omega} + \frac{1-j\omega}{1+\omega^2} - \frac{1-j\omega}{1+\omega^2} e^{-1-j\omega} \right\}$$

$$= \frac{1}{2} \left\{ \frac{2}{1+\omega^2} - \frac{2e^{-1}}{1+\omega^2} (\cos\omega - \omega \sin\omega) \right\}$$

$$= \boxed{\frac{1}{1+\omega^2} - \frac{e^{-1}}{1+\omega^2} (\cos\omega - \omega\sin\omega)}$$

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This sheet is for Question 2.

(b) $x(t) = \frac{1}{2}e^{j5t} + \frac{1}{2}e^{-j5t} + 3e^{j2\pi t}$

For $\omega = 5$, $H(j\omega) = \frac{1}{26} - \frac{e^{-1}}{26}\left(\cos(5) - 5\sin(5)\right)$

For $\omega = -5$, $H(j\omega) = \frac{1}{26} - \frac{e^{-1}}{26}\left(\cos(-5) + 5\sin(-5)\right)$

$$= \frac{1}{26} - \frac{e^{-1}}{26}\left(\cos(5) - 5\sin(5)\right)$$

For $\omega = 2\pi$, $H(j\omega) = \frac{1}{1+4\pi^2} - \frac{e^{-1}}{1+4\pi^2}\left(\cos(2\pi) - 2\pi \cancelto{0}{\sin(2\pi)}\right)$

$$= \frac{1}{1+4\pi^2}\left(1 - e^{-1}\right)$$

$$y(t) = \sum_{\omega} A_\omega e^{j\omega t} \cdot H(j\omega)$$

$$= \frac{1}{2}e^{j5t}\left[\frac{1}{26} - \frac{e^{-1}}{26}\left(\cos(5) - 5\sin(5)\right)\right]$$

$$+ \frac{1}{2}e^{-j5t}\left[\frac{1}{26} - \frac{e^{-1}}{26}\left(\cos(5) - 5\sin(5)\right)\right]$$

$$+ 3e^{j2\pi t}\frac{1}{1+4\pi^2}\left(1 - e^{-1}\right)$$

$$= \boxed{\frac{1}{26}\cos(5t)\left[1 - e^{-1}\left(\cos(5) - 5\sin(5)\right)\right] + \frac{3(1-e^{-1})}{1+4\pi^2}e^{j2\pi t}}$$

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Question 3: [22%] Consider a system with input-output relationship

$$y[n+2] = \alpha^{|n|}x[n+2]x[n]$$

for some constant α.

(a) [6%] Is the system time invariant? Prove or disprove it.

(b) [5%] Is the system causal? Is it memoryless?

(c) [5%] Is the system invertible? Prove it or provide a counter example.

(d) [6%] Is the system stable for $\alpha = -0.5$? How about for $\alpha = 2$? Explain.

(a)

Time shift at input : Let $x_1[n] = x[n-n_0]$

$$y_1[n] = \alpha^{|n-2|} x_1[n]\, x_1[n-2]$$
$$= \alpha^{|n-2|} x[n-n_0]\, x[n-n_0-2]$$

Time shift at output : $y[n-n_0] = \alpha^{|n-n_0-2|} x[n-n_0]\, x[n-n_0-2]$

If $\alpha = 1$, the system is time invariant since $y_1[n] = y[n-n_0]$

If $\alpha \neq 1$, the system is time variant since $y_1[n] \neq y[n-n_0]$

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This sheet is for Question 3.

(b)
Yes, the system is causal.
$\because$ the output only depends on the present and the past input,

No, the system has memory.
$\because$ the output also depends on the input in the past.

(c) No, the system is NOT invertible.
A counter example is as follows:
Consider $x_1[n] = 1 \quad \forall n$
and $x_2[n] = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$,

$x_1[n]$ and $x_2[n]$ give the same output $y[n]$.

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This sheet is for Question 3.

(d) For $\alpha = -0.5$, the system is stable.

$\because$ For any bounded input $|x[n]| \leq B \ \forall n$, where $B < \infty$ is a constant,

the output $|y[n+2]| = |\alpha^{|n|} x[n+2] x[n]|$

$$= |\alpha^{|n|}| \, |x[n+2]| \, |x[n]|$$

$$\leq 1 \cdot B \cdot B = B^2 \text{ is bounded.}$$

For $\alpha = 2$, the system is NOT stable.

$\because$ For any bounded input $|x[n]| \leq B \ \forall n$,

the output $|y[n+2]| \to \infty$ when $n \to \infty$.

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Question 4: [18%] Consider an LTI system with input-output relationship

$$y(t) = \int_t^{\infty} x(\tau)d\tau.$$

(a) [8%] What is the impulse response $h(t)$ of the system?

(b) [5%] Use $h(t)$ to prove whether the system is causal or not.

(c) [5%] Use $h(t)$ to prove whether the system is stable or not.

[Hint: If you are unsure how to use $h(t)$ for (b) and (c), you can answer by reasoning directly from the input-output relationship to obtain up to 3% on each part.]

(a) Let $x(t) = \delta(t)$,

$$h(t) = \int_t^{\infty} \delta(\tau)d\tau = \boxed{u(-t) \ \#} \quad \forall t.$$

(b) $\because$ $h(t) \neq 0$ when $t < 0$

$\therefore$ the system is NOT causal. #

(c) If a system is stable, $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$.

However, for this system,

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{0} 1\, dt \rightarrow \infty.$$

$\therefore$ the system is NOT stable. #

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This sheet is for Question 4.

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This sheet is for Question 4.

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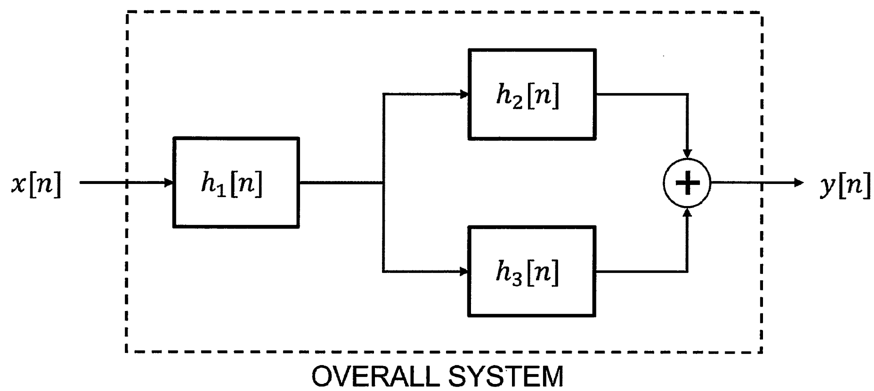
Question 5: [16%] Consider a DT system comprised of three LTI sub-systems with impulse responses $h_1[n]$, $h_2[n]$, and $h_3[n]$ connected in the configuration shown below. The three sub-systems have the following impulse responses:

$$h_1[n] = \delta[n-2] \qquad h_2[n] = u[n+2] \qquad h_3[n] = -u[n-2]$$



(a) [7%] Give an expression for the impulse response $h[n]$ of the overall system.
[Hint: Solve this question analytically, not graphically.]

(b) [3%] Sketch $h[n]$ from (a).

(c) [6%] Give an expression for the output $y[n]$ of the overall system when the input is:

$$x[n] = \delta[n-2] - \delta[n+4].$$

[Hint: If you are unsure how to answer (c), you can compute $x[n] * h_1[n]$ to receive up to 4%.]

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This sheet is for Question 5.

(a)

$$h[n] = h_1[n] * (h_2[n] + h_3[n])$$
$$= \delta[n-2] * (u[n+2] - u[n-2])$$
$$= \boxed{u[n] - u[n-4]}$$

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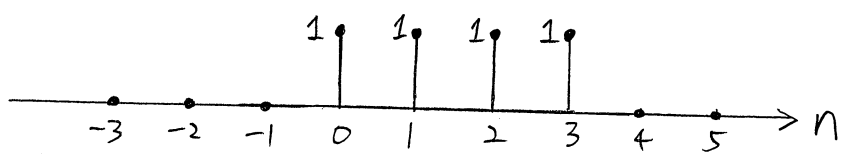
(b)



(c)

$$y[n] = x[n] * h[n]$$
$$= (\delta[n-2] - \delta[n+4]) * h[n]$$
$$= h[n-2] - h[n+4]$$
$$= (u[n-2] - u[n-6]) - (u[n+4] - u[n])$$
$$= \boxed{-u[n+4] + u[n] + u[n-2] - u[n-6]}$$

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This sheet is for Question 5.

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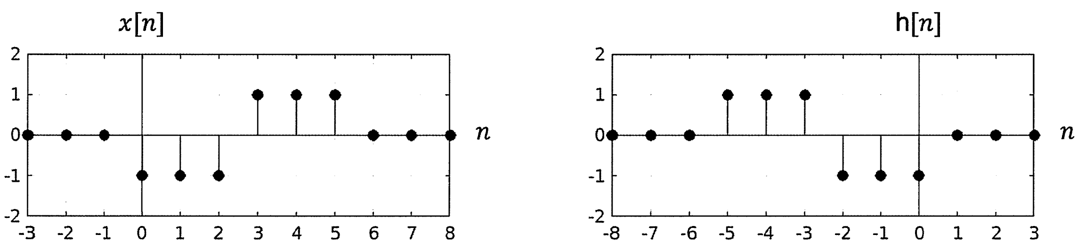
[Hint: This is a reasonably time consuming question that is worth only 10 points. The recommendation is to start working on this after you have finished the other questions to a reasonable degree.]

Question 6: [10%] Consider a discrete-time LTI system with an impulse response $h[n]$ shown on the right below. Suppose we input to this system the signal $x[n]$ shown on the left (note that $h[n] = x[-n]$).



(a) [7%] Determine the output $y[n]$ of the system.

(b) [3%] Sketch a plot of $y[n]$.

(a) $x[n] = -\delta[n] - \delta[n-1] - \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5]$

$$y[n] = x[n] * h[n]$$

$$= -(h[n] + h[n-1] + h[n-2]) + (h[n-3] + h[n-4] + h[n-5])$$ #

$$= \begin{cases} 6 & \text{if } n = 0 \\ 3 & \text{if } |n| = 1 \\ 0 & \text{if } |n| = 2 \\ -3 & \text{if } |n| = 3 \\ -2 & \text{if } |n| = 4 \\ -1 & \text{if } |n| = 5 \\ 0 & \text{if } |n| \geq 6 \end{cases}$$ #.

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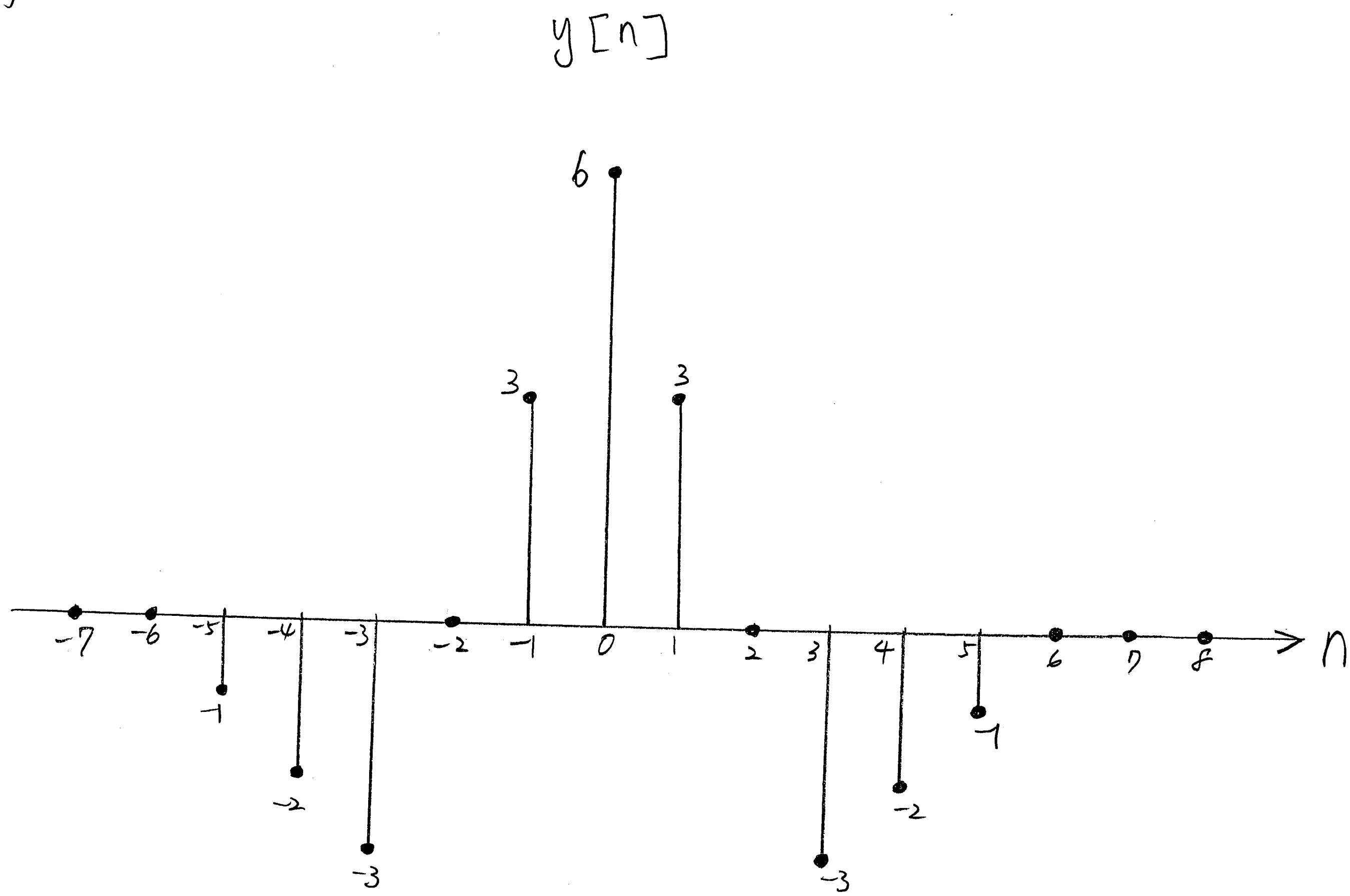
This sheet is for Question 6.

(b)



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This sheet is for Question 6.