ECE 301-001 and 301-003, Midterm #1
8–9:30pm, Wednesday, February 8, 2023, CL50 Rm224.

1. **Do not write answers on the back of pages!**
2. **After the exam ended, you will have 5 additional minutes to write down your name and Purdue ID on each of the pages.**
3. **If you need additional sheets to write down your answers, please let the instructor/TA know. We will hand out additional answer sheets then.**
4. Enter your student ID number, and signature in the space provided on this page.
5. This is a closed book exam.
6. This exam contains multiple-choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have **90 minutes** to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.
7. The instructor/TA will hand out loose sheets of paper for the rough work.
8. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

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Question 1: [14%, Energy and power]

Consider the following signal:

$$x[n] = \begin{cases} e^{-(n-3)}(\cos(0.25\pi n) + j\sin(0.25\pi n)) & \text{if } n \geq 3 \\ e^{n-3}(\cos(0.25\pi n) + j\sin(0.25\pi n)) & \text{if } n \leq 2 \end{cases}$$

(a) What is the total energy of $x[n]$?

(b) What is the overall average power of $x[n]$?

Hint: If $|r| < 1$, then we have the following formulas for computing the infinite sum of a geometric sequence:

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

$$\sum_{k=1}^{\infty} kar^{k-1} = \frac{a}{(1-r)^2}.$$

If $r \neq 1$, then we have the following formula for computing the finite sum of a geometric sequence:

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}.$$

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This sheet is for Question 1.

(a)

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{2} \left| e^{n-3} \left(\cos\frac{\pi}{4}n + j\sin\frac{\pi}{4}n\right)\right|^2 + \sum_{n=3}^{\infty} \left| e^{-(n-3)} \left(\cos\frac{\pi}{4}n + j\sin\frac{\pi}{4}n\right)\right|^2$$

$$= \sum_{n=-\infty}^{2} \left| e^{n-3} e^{j\frac{\pi}{4}n} \right|^2 + \sum_{n=3}^{\infty} \left| e^{-(n-3)} e^{j\frac{\pi}{4}n} \right|^2 \qquad \left(\begin{array}{c}\text{Note that} \\ |e^{j\theta}| = 1\end{array}\right)$$

$$= \sum_{n=-\infty}^{2} e^{2(n-3)} + \sum_{n=3}^{\infty} e^{-2(n-3)}$$

$$\sum_{n=-\infty}^{2} e^{2(n-3)} = \sum_{n=-\infty}^{-1} e^{2n} = \sum_{n=1}^{\infty} e^{-2n} = \sum_{n=1}^{\infty} e^{-2} e^{-2(n-1)} = \frac{e^{-2}}{1-e^{-2}}$$

$$\sum_{n=3}^{\infty} e^{-2(n-3)} = \sum_{n=1}^{\infty} e^{-2(n-1)} = \frac{1}{1-e^{-2}}$$

$$\therefore\ E = \sum_{n=-\infty}^{2} e^{2(n-3)} + \sum_{n=3}^{\infty} e^{-2(n-3)}$$

$$= \frac{e^{-2}}{1-e^{-2}} + \frac{1}{1-e^{-2}} = \frac{1+e^{-2}}{1-e^{-2}} \quad \#$$

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This sheet is for Question 1.

(b)

Since the signal has finite total energy,

the overall average power $= 0$ #

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \left\{ \sum_{n=-N}^{2} \left| e^{n-3} e^{-j\frac{\pi}{4}n} \right|^2 + \sum_{n=3}^{N} \left| e^{-(n-3)} e^{-j\frac{\pi}{4}n} \right|^2 \right\}$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \left\{ \sum_{n=-N}^{2} e^{2(n-3)} + \sum_{n=3}^{N} e^{-2(n-3)} \right\}$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \left\{ \frac{e^{-2}\left(1-\cancelto{0}{(e^{-2})^{N+3}}\right)}{1-e^{-2}} + \frac{e^{0}\left(1-\cancelto{0}{(e^{-2})^{N-2}}\right)}{1-e^{-2}} \right\}$$

$= 0$ #

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Question 2: [15%, Linearity]

Determine whether the following systems are linear or not. Be sure to justify your reasoning in each case.

(a) The system with input $x(t)$ and output

$$y(t) = \int_{\tau=0}^{10} x(t-\tau)u(\tau)d\tau,$$

where $u(t)$ is the unit step signal.

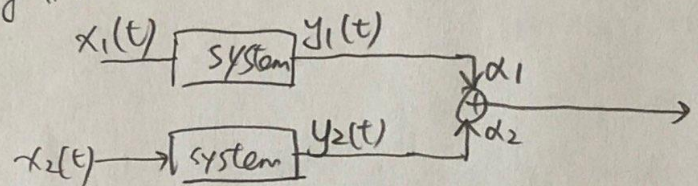
(b) The system with input $x[n]$ and output

$$y[n] = e^{x[n]}.$$

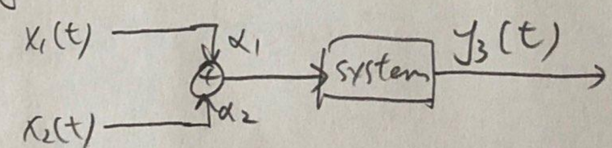
This sheet is for Question 2.

Definition: A system is linear if the output of the following two configurations are always identical.

Config #1



Config #2



(a) $$y(t) = \int_{\tau=0}^{10} x(t-\tau)u(\tau)\,d\tau = \int_{\tau=0}^{10} x(t-\tau)\,d\tau$$

$$y_1(t) = \int_{\tau=0}^{10} x_1(t-\tau)\,d\tau, \quad y_2(t) = \int_{\tau=0}^{10} x_2(t-\tau)\,d\tau$$

$$y_3(t) = \int_{\tau=0}^{10} \left[\alpha_1 x_1(t-\tau) + \alpha_2 x_2(t-\tau)\right] d\tau$$

$$= \alpha_1 \int_{\tau=0}^{10} x_1(t-\tau)\,d\tau + \alpha_2 \int_{\tau=0}^{10} x_2(t-\tau)\,d\tau$$

$$= \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

$\therefore$ linear #

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This sheet is for Question 2.

(b)

$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n] = e^{x[n]}$

$y_1[n] = e^{x_1[n]}, \quad y_2[n] = e^{x_2[n]}$

$$y_3[n] = e^{\alpha_1 x_1[n] + \alpha_2 x_2[n]} = \left(e^{x_1[n]}\right)^{\alpha_1} \cdot \left(e^{x_2[n]}\right)^{\alpha_2}$$

$$= \left(y_1[n]\right)^{\alpha_1} \cdot \left(y_2[n]\right)^{\alpha_2}$$

$$\neq \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

$\therefore$ nonlinear #

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Question 3: [14%, Algebra of signals]

Consider two signals

$$
\begin{aligned}
x(t) &= e^{j\omega t} \\
h(t) &= u(t+1) - u(t-2)
\end{aligned}
$$

(a) Find mathematical expressions for the signals $\frac{dx(t)}{dt}$ and $\frac{dh(t)}{dt}$.

(b) Determine $y(t)$ for a system that takes $x(t)$ as input and has the input-output relationship

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

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This sheet is for Question 3.

(a) $\frac{dx(t)}{dt} = \frac{d}{dt} e^{j\omega t} = j\omega e^{j\omega t}$ #

$\frac{dh(t)}{dt} = \frac{d}{dt}\left(u(t+1) - u(t-2)\right) = \delta(t+1) - \delta(t-2)$ #

(Note that $\delta(t) = \frac{d}{dt} u(t)$)

(b)

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$= \int_{\tau=-\infty}^{\infty} e^{j\omega\tau} [u(t-\tau+1) - u(t-\tau-2)] d\tau$$

$$= \int_{\tau=t-2}^{t+1} e^{j\omega\tau} d\tau$$

$$= \left[\frac{1}{j\omega} e^{j\omega\tau}\right]_{\tau=t-2}^{t+1}$$

$$= \frac{1}{j\omega}\left[e^{j\omega(t+1)} - e^{j\omega(t-2)}\right]$$

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This sheet is for Question 3.

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Question 4: [15%, HRCEs]

Recall that a family of discrete-time harmonically-related complex exponentials (DT HRCEs) is given by:

$$x_k[n] = e^{j\frac{2\pi k}{N}n}, \quad k = 0, ..., N-1$$

and a family of continuous-time HRCEs (CT HRCEs) is given by:

$$x_k(t) = e^{j\frac{2\pi k}{T}t}, \quad k = 0, \pm 1, \pm 2, ...$$

(a) Consider a fundamental frequency $\omega_0 = \frac{\pi}{8}$.

i. Write the expression for the DT HRCE signal family. What is the value of N?

ii. Which signal index k in the DT HRCE family has the highest rate of oscillation? Write the formula for this signal $x_k[n]$ in its most simplified form.

iii. Does the index k identified in (a)-ii also have the highest rate of oscillation for the corresponding CT HRCE family? Explain your answer.

(b) Next, consider a fundamental frequency $\omega_0 = \frac{1}{4}$.

i. Does the CT HRCE family exist? If so, write its expression and determine the value of T. Explain your answer.

ii. Does the DT HRCE family exist? If so, write its expression and determine the value of N. Explain your answer.

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This sheet is for Question 4.

(a)

(i) $N = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/8} = 16$ #

$x_k[n] = e^{j\frac{2\pi k}{16}n} = e^{j\frac{\pi k}{8}n}, \quad k = 0, 1, \ldots, 15$ #

(ii) $k = \frac{N}{2} = \frac{16}{2} = 8$ #

$x_8[n] = e^{j\pi n} = (-1)^n$ #

(iii) NO, for CT HRCE family, the rate of oscillation increases as the value of $|k|$ increases.

(b)

(i) Yes! $T = \frac{2\pi}{1/4} = 8\pi$ #

$x_k(t) = e^{j\frac{2\pi k}{8\pi}t} = e^{j\frac{k}{4}t}, \quad k = 0, \pm 1, \pm 2, \ldots$ #

The CT HRCE family exits because t can be any real number so that T can be any real positive number.

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This sheet is for Question 4.

(ii)

No! Because $x_k[n]$ only samples at integer points,

i.e., n only takes integer values, the period

$$N = LCM\left(\frac{2\pi}{1/4}, 1\right) = LCM(8\pi, 1)$$

does not exist.

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Question 5: [15%, Time transformations]

Consider the two signals, $x_1(t)$ and $x_2[n]$, as pictured below.

(a) Sketch a plot of $x_1(-\frac{1}{2}(t-1))$.

(b) Sketch a plot of $x_2[-2+2n]$.

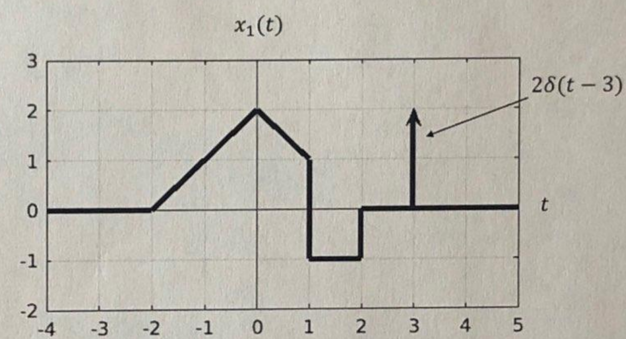

Figure 1: Plot of $x_1(t)$.

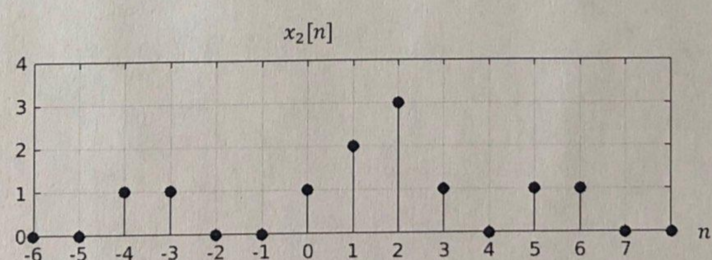

Figure 2: Plot of $x_2[n]$.

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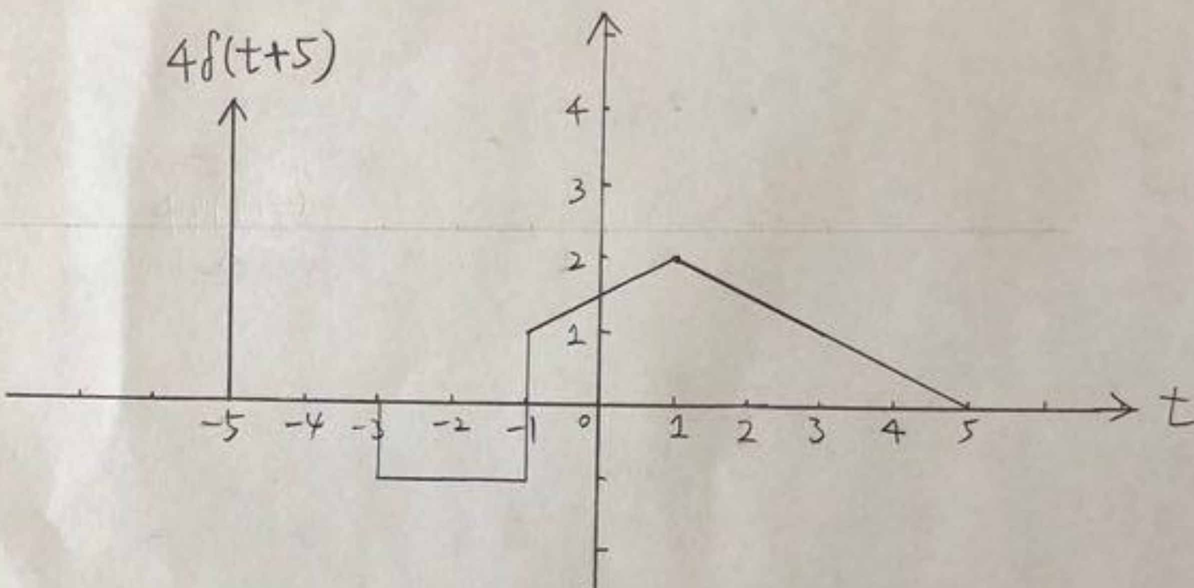
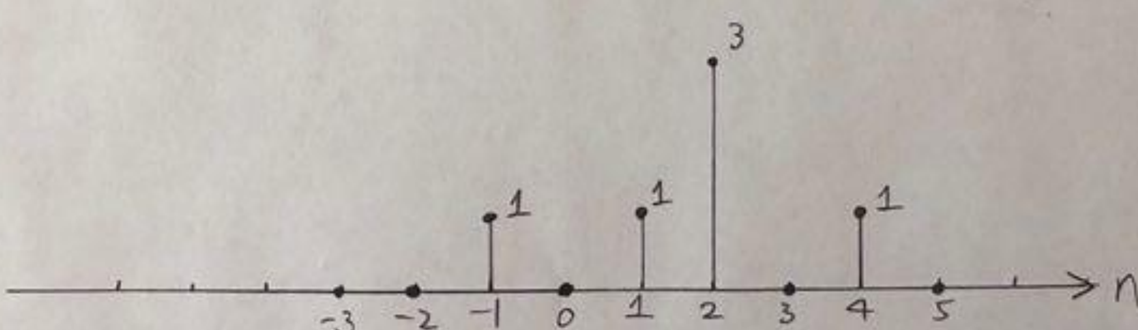
This sheet is for Question 5.

(a) $x_1\left(-\frac{1}{2}(t-1)\right) = x_1\left(-\frac{1}{2}t+\frac{1}{2}\right)$



(b) $x_2[-2+2n]$



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This sheet is for Question 5.

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Question 6: [14%, Periodicity]

(a) Is the signal

$$x(t) = \sum_{k=-\infty}^{\infty} (\delta(t-4k) + \delta(t-4k-1)) + e^{j\frac{3\pi}{2}t} + \sin\left(\frac{\pi}{3}t + \frac{\pi}{6}\right)$$

periodic? If so, determine its fundamental period.

(b) Repeat (a) for the signal

$$x[n] = \cos\left(\frac{n-\pi}{5}\right) + e^{j6\pi n}.$$

(a)

signal	period
$\delta(t-4k)$	4
$\delta(t-4k-1)$	4
$e^{j\frac{3\pi}{2}t}$	$\frac{2\pi}{3\pi/2} = \frac{4}{3}$
$\sin(\frac{\pi}{3}t+\frac{\pi}{6})$	$\frac{2\pi}{\pi/3} = 6$

Since $\delta(t-4k)$, $\delta(t-4k-1)$, $e^{j\frac{3\pi}{2}t}$ and $\sin(\frac{\pi}{3}t+\frac{\pi}{6})$ are all periodic, the signal $x(t)$ is periodic with fundamental period $= LCM(4, 4, \frac{4}{3}, 6) = 12$ #

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This sheet is for Question 6.

(b)

signal period

$\cos\left(\frac{n-\pi}{5}\right)$ $LCM\left(\frac{2\pi}{1/5}, 1\right) = LCM(10\pi, 1)$ does not exist.

$e^{j6\pi n}$ $LCM\left(\frac{2\pi}{6\pi}, 1\right) = LCM\left(\frac{1}{3}, 1\right) = 1$

$\therefore$ $x[n]$ is aperiodic.

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This sheet is for Question 6.

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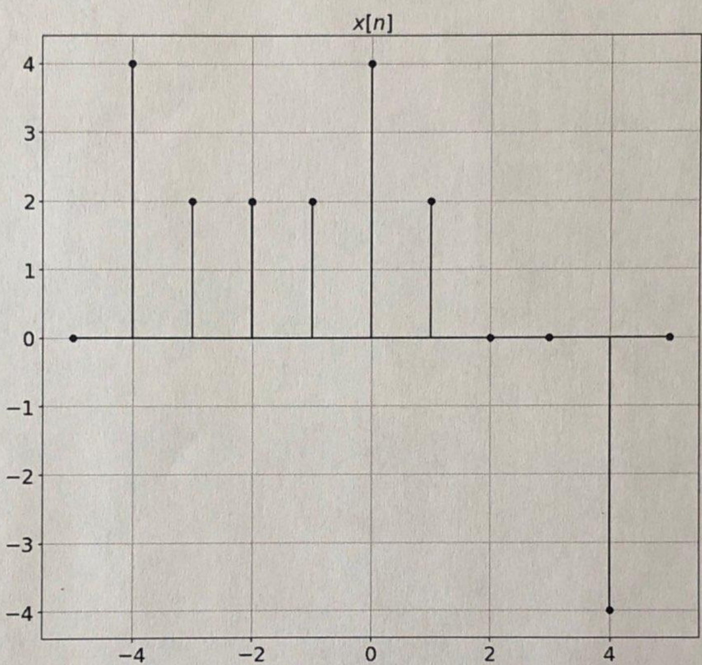
Question 7: [13%, Even-odd decomposition]

Consider the signal $x[n]$ depicted below.



(Note that $x[n] = 0$ for all $|n| \geq 5$)

(a) Is $x[n]$ even, odd, or neither?

(b) Decompose $x[n] = x_{even}[n] + x_{odd}[n]$ into its even and odd signal components. Give plots of $x_{even}[n]$ and $x_{odd}[n]$.

(a) Neither

(b) $x_{even}[n] = \dfrac{x[n] + x[-n]}{2}$

$x_{odd}[n] = \dfrac{x[n] - x[-n]}{2}$

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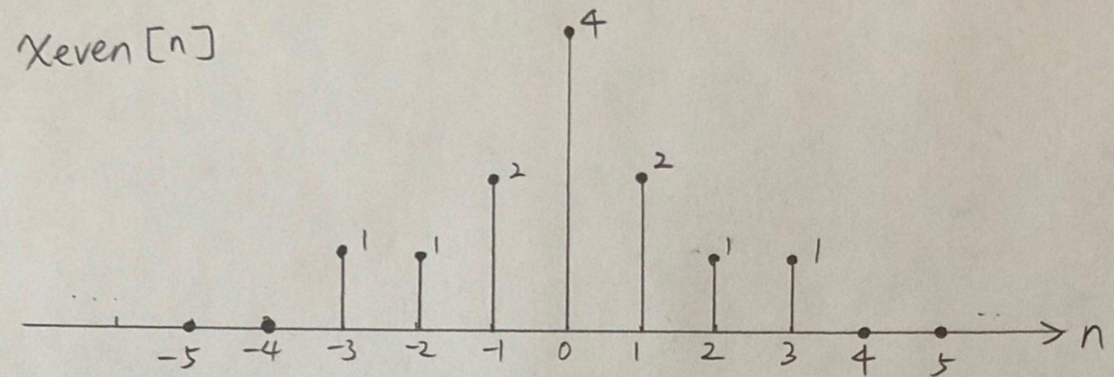
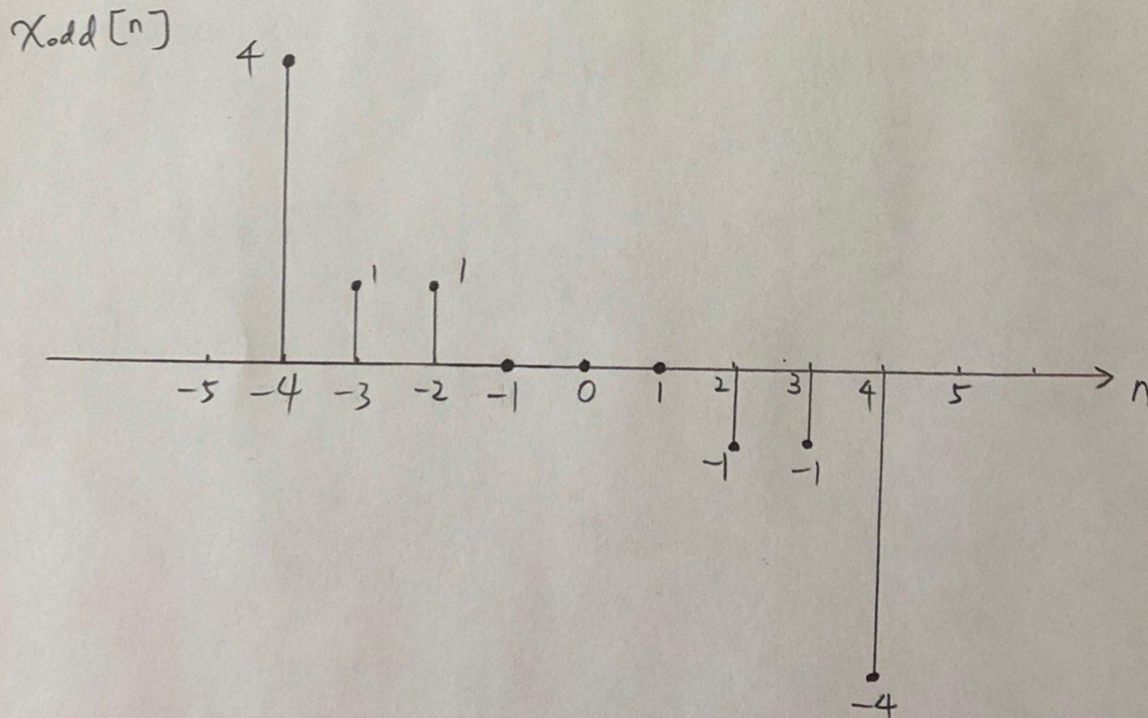
This sheet is for Question 7.





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