ECE 301-001&003, Final Exam
1–3pm, Thursday, May 4, 2023, PHYS 203 and BHEE 170.

1. **Do not write answers on the back of pages!**
2. **After the exam ends, you will have 5 additional minutes to write down your name and Purdue ID on each of the pages.**
3. **If you need additional sheets of paper to write down your answers, please let one of the proctors know. We will hand out additional answer sheets as needed.**
4. Write your student ID number and signature in the space provided on this page.
5. This is a closed book exam. Neither calculators nor help sheets are allowed. A separate formula packet has been provided to you.
6. You have **120 minutes** to complete the exam. There are 8 multi-part questions.
7. You must **show all work** used to arrive at your answer. This is required to receive full credit, and also is helpful for you in getting partial credit.

Name:

Student ID:

As a Boilermaker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date: 2023/5/4

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Question 1: [12%]

Consider a continuous time system with input $x(t)$ and output $y(t)$ related as:

$$y(t) = \int_{s=-\infty}^{t} x(s-\pi)e^{(3+j)(t-s)}ds$$

(a) [4%] Is the above system linear? Please carefully justify your answer. A correct answer without any justification will receive only 1.5 points.

(b) [3%] Is the above system causal? Please carefully justify your answer. A correct answer without any justification will receive only 1.5 points.

(c) [5%] Find the expression of the impulse response of the above system.

(a)

Let $y_1(t) = \int_{s=-\infty}^{t} x_1(s-\pi)e^{(3+j)(t-s)}ds$

$y_2(t) = \int_{s=-\infty}^{t} x_2(s-\pi)e^{(3+j)(t-s)}ds$

and $x_3(t) = ax_1(t) + bx_2(t)$ where a and b are any complex constant.

$$y_3(t) = \int_{s=-\infty}^{t} x_3(s-\pi)e^{(3+j)(t-s)}ds = \int_{s=-\infty}^{t}\left(ax_1(s-\pi)+bx_2(s-\pi)\right)e^{(3+j)(t-s)}ds$$

$$= a\int_{s=-\infty}^{t} x_1(s-\pi)e^{(3+j)(t-s)}ds + b\int_{s=-\infty}^{t} x_2(s-\pi)e^{(3+j)(t-s)}ds$$

$$= ay_1(t) + by_2(t)$$

$\therefore$ The system is linear. #

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This sheet is for Question 1.

(b) Yes, the system is causal because the output $y(t)$ depends only on the present and/or the past input.

Speeifically, since the integration is over the range of $s<t$ and the input $x(s-\pi)$ inside the integral will have the maximum range being $x(t-\pi)$. Therefore, it won't depend on the future, say something like $x(t+1)$.

(c) Note that when $x(t)=\delta(t)$, $y(t)=h(t)$.

$$\therefore h(t) = \int_{s=-\infty}^{t} \delta(s-\pi)\, e^{(3+j)(t-s)}\, ds$$

$$= \begin{cases} 0 & \text{when } t<\pi \\ \int_{s=-\infty}^{t} \delta(s-\pi)\, e^{(3+j)(t-s)}\, ds & \text{when } t\geq\pi \end{cases}$$

$$= \boxed{e^{(3+j)(t-\pi)}\, u(t-\pi)}$$

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This sheet is for Question 1.

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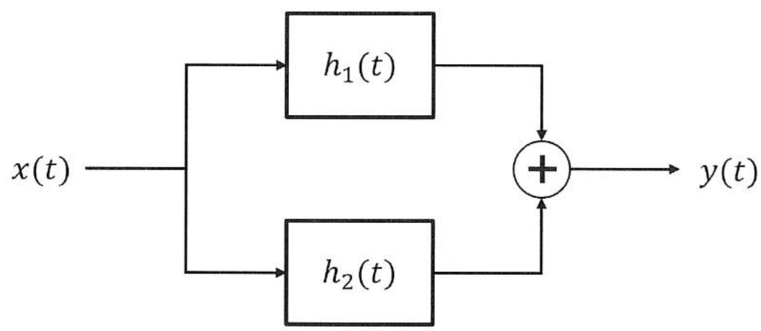
Question 2: [11.5%]

Consider the CT-LTI system shown in the figure below, which is comprised of two LTI sub-systems with impulse responses $h_1(t)$ and $h_2(t)$:



Let us define $h_1(t)$ and $h_2(t)$ as follows:

$$h_1(t) = \begin{cases} 2 & -1 \leq t < 0 \\ 0 & \text{else} \end{cases}$$

$$h_2(t) = \begin{cases} 2 & 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$

(a) [7%] Find the output of the system, $y(t)$, when the input $x(t)$ is defined as:

$$x(t) = \begin{cases} t & -1 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$

[Hint: You may find it useful to first specify the impulse response of the overall system.]

(b) [4.5%] With $x(t)$ from part (a), sketch a plot of the signal $z(t)$ defined as

$$z(t) = x(-2t + 1) + 1$$

for the range $-2 \leq t \leq 2$.

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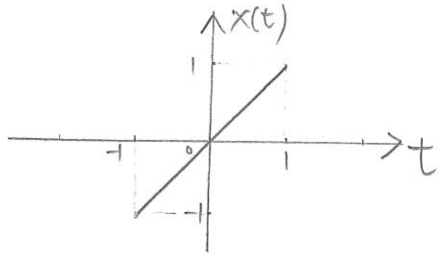
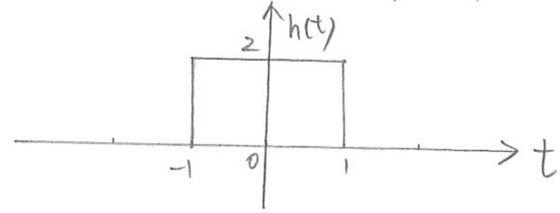
This sheet is for Question 2.

(a)

The overall system $h(t) = h_1(t) + h_2(t) = \begin{cases} 2, & -1 \le t \le 1 \\ 0, & \text{else} \end{cases}$





$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

1° when $t < -2$, $y(t) = 0$

2° when $-2 \le t < 0$, $y(t) = \int_{-1}^{t+1} 2\tau\, d\tau = \tau^2 \Big|_{-1}^{t+1}$

$$= (t+1)^2 - 1 = t^2 + 2t$$

3° when $0 \le t < 2$, $y(t) = \int_{t-1}^{1} 2\tau\, d\tau = \tau^2 \Big|_{t-1}^{1}$

$$= 1 - (t-1)^2 = -t^2 + 2t$$

4° when $t \ge 2$, $y(t) = 0$.

$$\boxed{y(t) = \begin{cases} t^2 + 2t & \text{when } -2 \le t < 0 \\ -t^2 + 2t & \text{when } 0 \le t < 2 \\ 0 & \text{else} \end{cases}}$$

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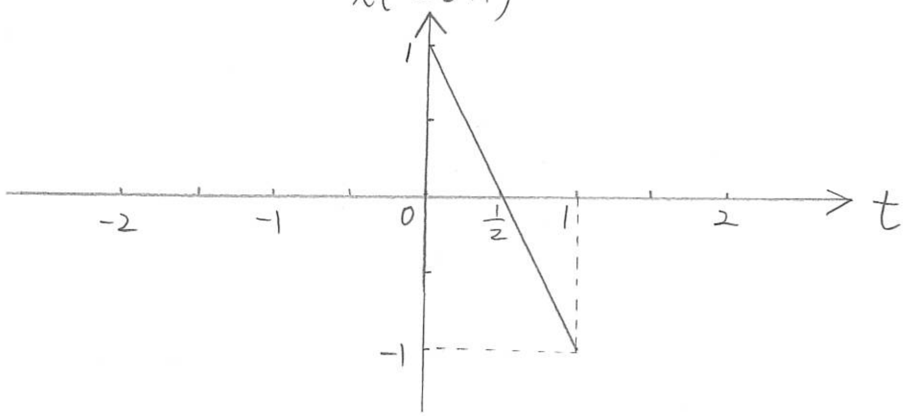
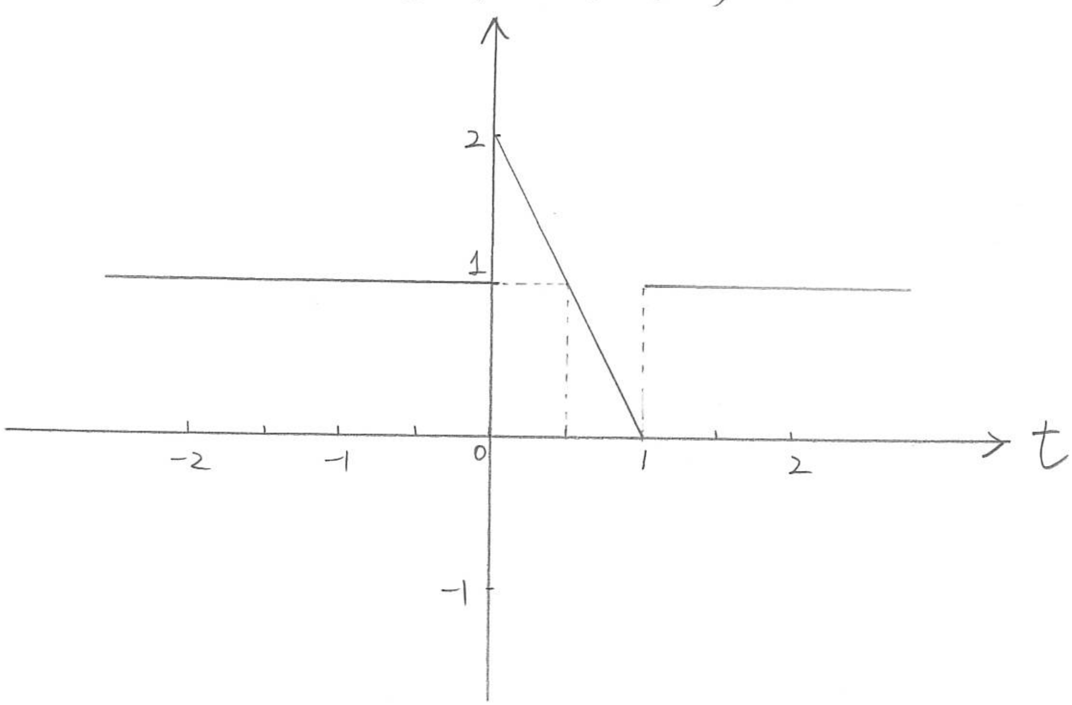
This sheet is for Question 2.

(b) Recall that if we want $cx(at+b)$,
the order of operations is b, a, c.

$x(-2t+1)$



$z(t) = x(-2t+1)+1$



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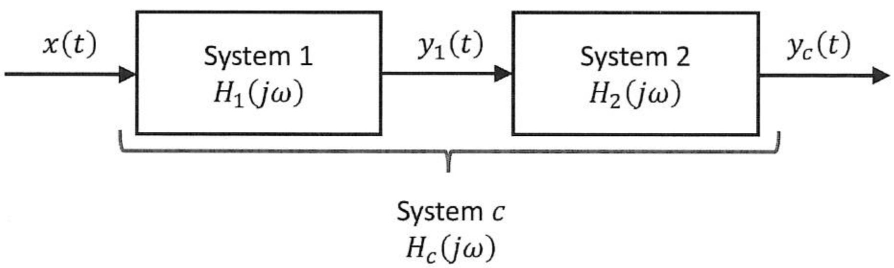
Question 3: [13%]

Consider the following cascade of two CT-LTI systems, with the composite system referred to as System c:



When we input

$$x(t) = te^{-4t}u(t)$$

to System 1, we observe an output

$$y_1(t) = t^2e^{-4t}u(t)$$

from System 1.

(a) [7%] Find the frequency response $H_1(j\omega)$.

(b) [3%] Suppose the frequency response of System 2 is

$$H_2(j\omega) = 10$$

Is System c invertible? Please carefully justify your answer.

(c) [3%] Repeat part (b) if

$$H_2(j\omega) = \sum_{k=-\infty}^{\infty} \delta(\omega - k\pi)$$

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This sheet is for Question 3.

(a) By TABLE 4.2,

$$X(j\omega) = \mathcal{F}\{x(t)\} = \frac{1}{(4+j\omega)^2}$$

$$Y_1(j\omega) = \mathcal{F}\{y_1(t)\} = \mathcal{F}\{2 \cdot \frac{t^{3-1}}{(3-1)!} e^{-4t} u(t)\}$$

$$= 2 \cdot \frac{1}{(4+j\omega)^3}$$

$$\boxed{H_1(j\omega) = \frac{Y_1(j\omega)}{X(j\omega)} = \frac{2}{4+j\omega}}$$

#

(b) $H_c(j\omega) = H_1(j\omega) H_2(j\omega) = \frac{20}{4+j\omega}$

Yes, System c is invertible because $H_c(j\omega) \neq 0$ for all ω.

(c) $H_c(j\omega) = H_1(j\omega) H_2(j\omega) = \frac{2}{4+j\omega}\left(\sum_{k=-\infty}^{\infty} \delta(\omega - k\pi)\right)$

No, System c is not invertible

because $H_c(j\omega) = 0$ when $\omega \neq k\pi$ for $k \in \mathbb{Z}$

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This sheet is for Question 3.

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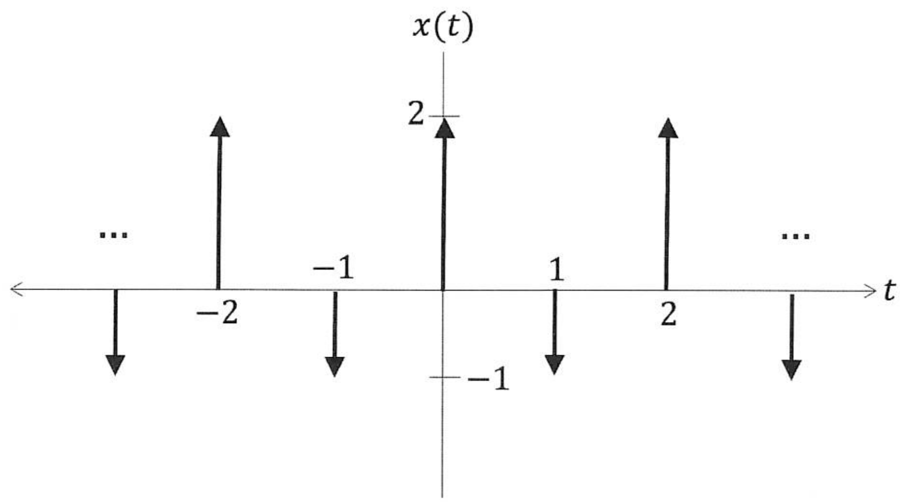
Question 4: [13%]

Consider the following periodic signal $x(t)$ consisting of impulses alternating between weights of 2 and -1:



(a) [2%] Is $x(t)$ even, odd, or neither? Please carefully justify your answer.

(b) [2%] Let $\{a_k\}$ denote the Fourier series coefficients of $x(t)$. Will the magnitude $|a_k|$ be even, odd, or neither? You must carefully justify your answer without performing any calculations.

(c) [2%] Again without performing any calculations, can we conclude that the phase $\angle a_k = 0$ for all k? Please carefully justify your answer.

(d) [7%] Compute $\{a_k\}$, and use them to write $x(t)$ as its Fourier series synthesis.

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This sheet is for Question 4.

(a) $x(t)$ is even $\because x(t) = x(-t)$

(b) Since $x(t)$ is even, $a_k = a_{-k}$. $\Rightarrow |a_k| = |a_{-k}|$

$\therefore |a_k|$ is even.

(c) Yes, $\measuredangle a_k = 0$ for all k $\because x(t)$ is real and even.

(d) $$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\left(\frac{2\pi}{T}\right)t}\,dt$$

$T = 2$.

For $k=0$, $$a_0 = \frac{1}{2}\int_{t=-\frac{1}{2}}^{3/2} \left(2\delta(t) - \delta(t-1)\right) \cancelto{1}{e^{-j0\pi t}}\,dt = \frac{1}{2}$$

For $k \neq 0$, $$a_k = \frac{1}{2}\int_{t=-\frac{1}{2}}^{3/2} \left(2\delta(t) - \delta(t-1)\right) e^{-jk\pi t}\,dt$$

$$= \frac{1}{2}\left(2 - e^{-jk\pi}\right) = \frac{1}{2}\left(2 - (-1)^k\right)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\pi t} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\left(2-(-1)^k\right)\right) e^{jk\pi t}$$

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This sheet is for Question 4.

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Question 5: [12%]
Consider a discrete time signal

$$x[n] = \begin{cases} e^{j\frac{14\pi}{3}n} + \delta[n] & \text{if } 0 \le |n| \le 3 \\ \text{periodic with period } 6 \end{cases}.$$

Let $\{a_k\}$ denote the DTFS coefficients of $x[n]$.

(a) [8%] Plot a_k for the range of $0 \le k \le 3$.

[Hint: Try breaking down $x[n] = x_1[n] + x_2[n]$ to solve this question.]

Consider a discrete-time ideal low-pass filter with cutoff frequency $W = 0.4\pi$ rad/sec. Namely, its frequency response satisfies

$$H(e^{j\omega}) = \begin{cases} 1 & \text{if } |\omega| \le 0.4\pi \\ 0 & \text{if } 0.4\pi < |\omega| \le \pi \end{cases}$$

Let $y[n]$ denote the output when feeding $x[n]$ through the above discrete-time low pass filter.

(b) [4%] Let $\{b_k\}$ denote the DTFS coefficients of the output $y[n]$. Answer the following yes/no questions,

i) Is $b_0 = 0$?

ii) Is $b_1 = 0$?

iii) Is $b_2 = 0$?

iv) Is $b_3 = 0$?

v) Is $b_4 = 0$?

vi) Is $b_5 = 0$?

and briefly explain your answers.

[Hint: If you do not know the answer to this subquestion, you can write down the relationship between b_k and a_k in terms of $H(e^{j\omega})$. You will receive 1.5 points for this subquestion if your answer is correct.]

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This sheet is for Question 5.

(a) Let $x_1[n] = e^{j\frac{14\pi}{3}n}$ and $x_2[n] = \delta[n]$.

Denote $\{c_k\}$ the DTFS coefficients of $x_1[n]$

and $\{d_k\}$ the DTFS coefficients of $x_2[n]$

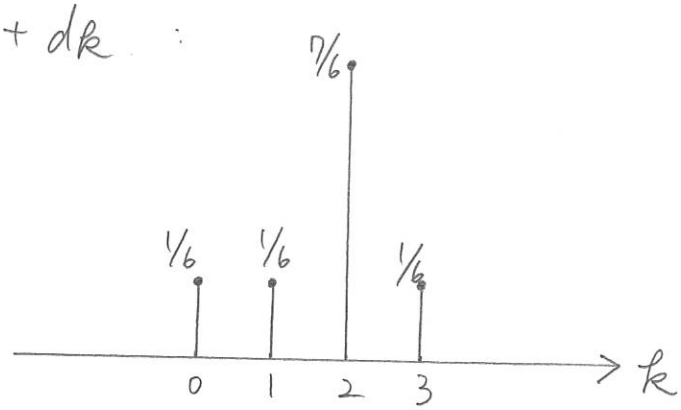
By TABLE 5.2,

for $x_1[n]$: $\omega_0 = \frac{14\pi}{3} = \frac{2\pi \cdot 14}{6}$

$$\Rightarrow c_k = \begin{cases} 1, & k = 14,\ 14 \pm 6,\ 14 \pm 12, \dots \\ 0, & \text{otherwise.} \end{cases}$$

for $x_2[n]$: $d_k = \frac{1}{6}$ for all k

$a_k = c_k + d_k$:



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This sheet is for Question 5.

(b) $b_k = a_k \cdot H(e^{jk\frac{2\pi}{6}})$

i) No, $\because b_0 = a_0 \cdot H(e^{j0}) \neq 0$

ii) No, $\because b_1 = a_1 \cdot H(e^{j\frac{\pi}{3}}) = a_1 \cdot 1 \neq 0$.

iii) Yes, $\because b_2 = a_2 \cdot H(e^{j\frac{2}{3}\pi}) = a_1 \cdot 0 = 0$.

iv) Yes, $\because b_3 = a_3 \cdot H(e^{j\pi}) = a_3 \cdot 0 = 0$

v) Yes, $\because b_4 = b_{-2} = a_{-2} \cdot H(e^{j(-\frac{2}{3})\pi}) = 0$

vi) No, $\because b_5 = b_{-1} = a_{-1} \cdot H(e^{j(-\frac{1}{3})\pi}) = a_{-1} \cdot 1 \neq 0$.

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Question 6: [13%]

Consider a causal DT-LTI system characterized by the difference equation

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 2x[n-1].$$

(a) [6%] Determine the frequency response $H(e^{j\omega})$ of the system.

(b) [4%] Determine the impulse response $h[n]$ of the system.

(c) [3%] Is the system stable? Justify your answer. A correct answer without any justification will receive only 1 point.

(a) $Y(e^{j\omega}) + \frac{1}{4}e^{-j\omega}Y(e^{j\omega}) - \frac{1}{8}e^{-j2\omega}Y(e^{j\omega})$

$= 2e^{-j\omega}X(e^{j\omega})$

$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-j2\omega}}$ #

Let $H(e^{j\omega}) = \frac{2e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-j2\omega}} = \frac{A}{(1-\frac{1}{4}e^{-j\omega})} + \frac{B}{(1+\frac{1}{2}e^{-j\omega})}$

$A(1+\frac{1}{2}e^{-j\omega}) + B(1-\frac{1}{4}e^{-j\omega}) = 2e^{-j\omega}$

Let $e^{-j\omega} = -2$, $\frac{3}{2}B = -4 \Rightarrow B = -\frac{8}{3}$

Let $e^{-j\omega} = 4$, $3A = 8 \Rightarrow A = \frac{8}{3}$

Hence, $H(e^{j\omega}) = \frac{\frac{8}{3}}{(1-\frac{1}{4}e^{-j\omega})} + \frac{-\frac{8}{3}}{(1+\frac{1}{2}e^{-j\omega})}$

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This sheet is for Question 6.

By TABLE 5.2,

$$h[n] = \frac{8}{3}\left(\frac{1}{4}\right)^n u[n] - \frac{8}{3}\left(-\frac{1}{2}\right)^n u[n]$$

\#

(c)

Yes, the system is stable.

$$\because \sum_{n=-\infty}^{\infty} |h[n]| = \frac{8}{3} \sum_{n=0}^{\infty} \left| \left(\frac{1}{4}\right)^n - \left(-\frac{1}{2}\right)^n \right|$$

$$< \frac{8}{3}\left(\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \right)$$

$$< \infty$$

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This sheet is for Question 6.

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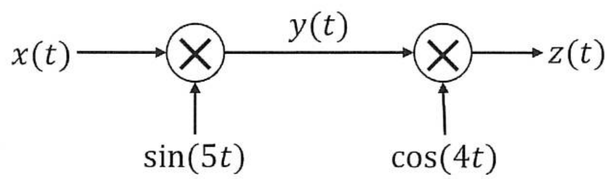
Question 7: [13.5%]



Suppose we feed the above system with the following continuous time signal $x(t)$:

$$x(t) = \sin(t)$$

(a) [2%] Plot $X(j\omega)$ for the range of $-12 < \omega < 12$. Please carefully mark both the horizontal and vertical axes of your figure. For simplicity, if the value $X(j\omega)$ is $3j$, then you can mark the vertical axis as $3j$.

[Hint 1: If you do not know how to plot $X(j\omega)$, you can simply find the expression of $X(j\omega)$. You will receive 1.5 points if your answer is correct.]

Define $y(t) = x(t) \cdot \sin(5t)$.

(b) [5%] Plot $Y(j\omega)$ for the range of $-12 < \omega < 12$. Please carefully mark both the horizontal and vertical axes of your figure, as in part (a).

[Hint 2: If you do not know how to solve this subquestion, please write down the relationship between $Y(j\omega)$ and $X(j\omega)$. You will receive 3 points if your answer is correct.]

Define $z(t) = y(t) \cdot \cos(4t)$.

(c) [6.5%] Plot $Z(j\omega)$ for the range of $-12 < \omega < 12$. Please carefully mark both the horizontal and vertical axes of your figure, as in part (a).

[Hint 3: If you do not know how to solve this subquestion, please solve the following question instead. Suppose $z_3(t) = (\sin(5t))^2$. Find out the corresponding Fourier transform $Z_3(j\omega)$. You will receive 5 points if your answer is correct.]

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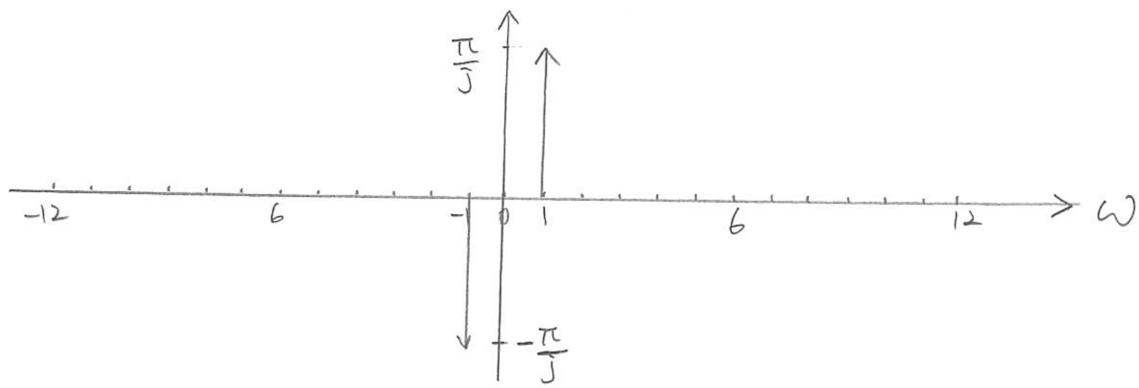
This sheet is for Question 7.

(a) By TABLE 4.2, $X(j\omega) = \frac{\pi}{j}[\delta(\omega-1) - \delta(\omega+1)]$

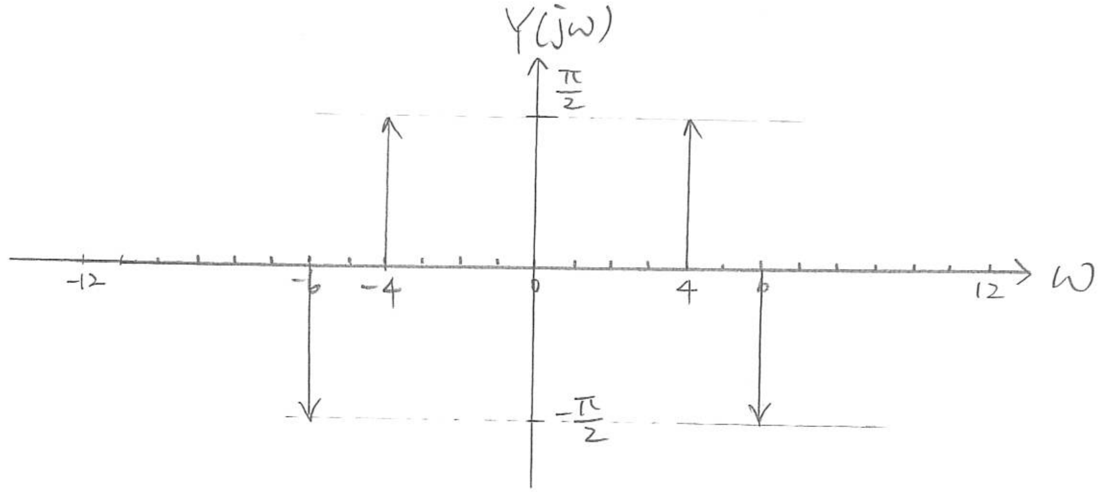


(b) $y(t) = x(t)\cdot\sin(5t) = x(t)\cdot\frac{1}{2j}(e^{j5t} - e^{-j5t})$

$Y(j\omega) = \frac{1}{2j}(X(j(\omega-5)) - X(j(\omega+5)))$

$= -\frac{\pi}{2}[\delta(\omega-6) - \delta(\omega-4) - \delta(\omega+4) + \delta(\omega+6)]$

$= \frac{\pi}{2}[-\delta(\omega-6) + \delta(\omega-4) + \delta(\omega+4) - \delta(\omega+6)]$



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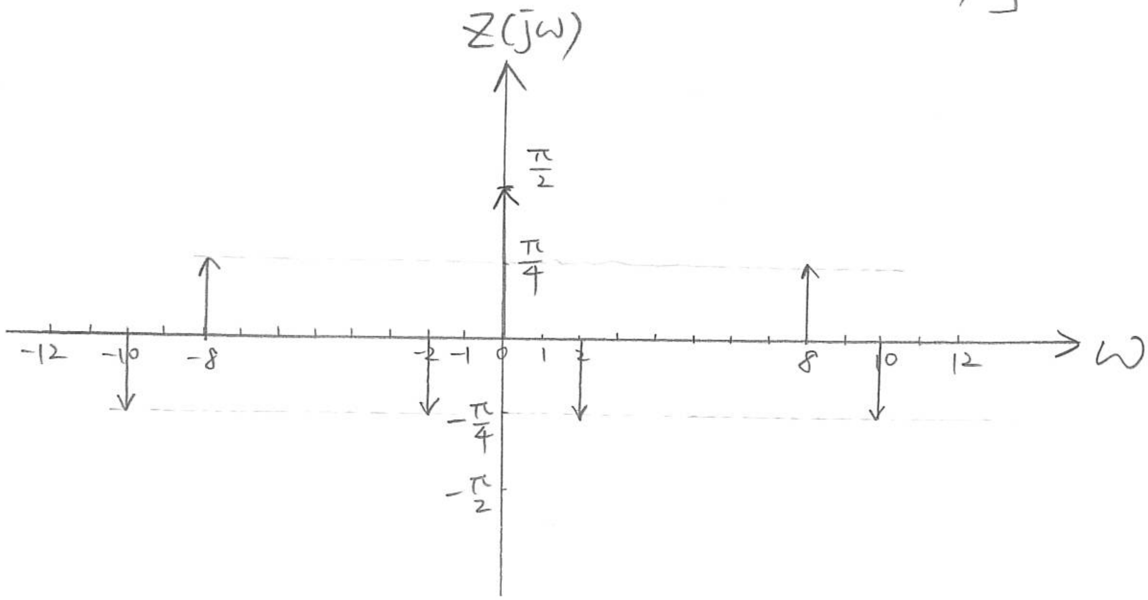
This sheet is for Question 7.

(c) $z(t) = y(t) \cdot \cos(4t) = y(t) \cdot \frac{1}{2}\left(e^{j4t} + e^{-j4t}\right)$

$$Z(j\omega) = \frac{1}{2}\Big(Y(j(\omega-4)) + Y(j(\omega+4)) \Big)$$

$$= \frac{\pi}{4}\Big[-\delta(\omega-10) + \delta(\omega-8) + \delta(\omega) - \delta(\omega+2) - \delta(\omega-2) + \delta(\omega) + \delta(\omega+8) - \delta(\omega+10) \Big]$$

$$= \frac{\pi}{4}\Big[-\delta(\omega-10) + \delta(\omega-8) - \delta(\omega-2) + 2\delta(\omega) - \delta(\omega+2) + \delta(\omega+8) - \delta(\omega+10) \Big]$$



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Question 8: [12%]

Let us define a DT signal $x[n]$ as

$$x[n] = e^{j\pi n} \cdot (u[n+2] - u[n-3])$$

(a) [6%] Find the DTFT $X(e^{j\omega})$.

(b) [6%] Find the value of the expression

$$\int_{\pi}^{3\pi} |X(e^{j\omega})|^2 d\omega$$

[Hint: Consider how this expression relates to total energy.]

(a) Let $x_1[n] = u[n+2] - u[n-3] = \begin{cases} 1, & |n| \leq 2 \\ 0, & |n| > 2 \end{cases}$

By TABLE 5.2,

$$X_1(e^{j\omega}) = \frac{\sin[\omega(2+\frac{1}{2})]}{\sin(\omega/2)} = \frac{\sin(\frac{5}{2}\omega)}{\sin(\frac{1}{2}\omega)}$$

Applying the frequency shifting property,

$$X(e^{j\omega}) = X_1(e^{j(\omega-\pi)}) = \boxed{\frac{\sin(\frac{5}{2}(\omega-\pi))}{\sin(\frac{1}{2}(\omega-\pi))}}$$

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This sheet is for Question 8.

(b) By Parseval's Relation for Aperiodic Signals

$$\int_{\pi}^{3\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= 2\pi \left(\sum_{n=-2}^{2} 1 \right) = \boxed{10\pi}$$

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This sheet is for Question 8.