PURDUE

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EXAM TITLE

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1. Make sure your name and PUID are clearly written at the top of every page, including any additional blank pages you use.
2. Write only on the front of the exam pages.
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5. Answer all questions in the area designated for that answer. Do not run over into the next question space.

Midterm #3 of ECE 301-004, (CRN: 13890)
8–9pm, Thursday, April 5, 2022, FRNY G140.

1. **<u>Do not write answers on the back of pages!</u>**
2. **After the exam ended, you will have 5 additional minutes to write down your name and Purdue ID on each of the pages.**
3. **If you need additional sheets to write down your answers, please let the instructor/TA know. We will hand out additional answer sheets then.**
4. Enter your student ID number, and signature in the space provided on this page.
5. This is a closed book exam.
6. This exam contains only work-out questions. You have one hour to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.
7. If needed and requested by students, the instructor/TA will hand out loose sheets of paper for the rough work.
8. Neither calculators nor help sheets are allowed.

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As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

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Question 1: [16%, Work-out question]

Consider the following periodic signal

$$x(t) = \begin{cases} t & \text{if } 0 \leq t < 3 \\ 3 & \text{if } 3 \leq t < 4 \\ 15 - 3t & \text{if } 4 \leq t < 5 \\ \text{periodic with period } T = 5 & \end{cases} \tag{1}$$

Define $y(t) = \frac{d}{dt}x(t)$.

1. [3%] Plot $y(t)$ for the range of $-5 \leq t \leq 5$.
2. [7%] Denote the CTFS of $y(t)$ by (b_k, ω_y). Find the b_4 value.

   Hint: Your answer can be of the following form:

   $$b_4 = \frac{1}{10}\left(\frac{e^{3.5\pi} - 1}{0.25\pi} + \frac{e^{5\pi} - 1}{0.5\pi}\right) \tag{2}$$

   There is no need to further simplify it.
3. [6%] Denote the CTFS of $x(t)$ by (a_k, ω_x). Find the value of a_4. and a_0

   Hint: If you do not know the answer to Q1.2, you can assume $b_k = ke^{-k\cdot(j+1)}$. **You will receive full credit of Q1.3** if your answer is correct.

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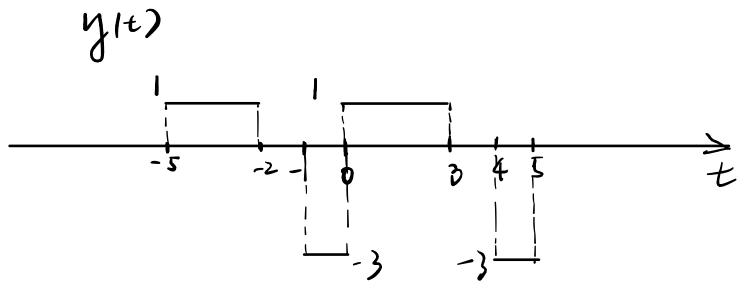
This sheet is for Question 1.

Answer:

1) $$y(t) = \begin{cases} 1 & 0 \le t < 3 \\ 0 & 3 \le t < 4 \\ -3 & 4 \le t < 5 \\ \text{periodic with } T=5. \end{cases}$$



2) $$b_4 = \frac{1}{T}\int_T y(t)\, e^{-j4\cdot\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{5}\int_0^3 e^{-j\frac{8\pi}{5}t}\, dt - \frac{3}{5}\int_4^5 e^{-j\frac{8\pi}{5}t}\, dt$$

$$= \frac{1}{5}\,\frac{1}{-j\frac{8\pi}{5}}\, e^{-j\frac{8\pi}{5}t}\Big|_0^3 - \frac{3}{5}\,\frac{1}{-j\frac{8\pi}{5}}\, e^{-j\frac{8\pi}{5}t}\Big|_4^5$$

$$= \frac{1}{-j8\pi}\left(e^{-j\frac{24\pi}{5}} - 1\right) - \frac{3}{-j8\pi}\left(1 - e^{-j\frac{32\pi}{5}}\right)$$

$$= \frac{1}{j2\pi} - \frac{1}{j8\pi}e^{-j\frac{24\pi}{5}} - \frac{3}{j8\pi}e^{-j\frac{32\pi}{5}}$$

Method 2:

Let $$y_1(t) = \begin{cases} 1 & |t| < 1.5 \\ 0 & \text{otherwise} \\ \text{periodic } T=5 \end{cases} \qquad y_2(t) = \begin{cases} -3 & |t| < 0.5 \\ 0 & \text{otherwise} \\ \text{periodic } T=5 \end{cases}$$

$$y(t) = y_1(t-1.5) - 3\, y_2(t-4.5)$$

$$b_k = \frac{\sin\left(4\cdot\frac{2\pi}{5}\,\frac{3}{2}\right)}{k\pi} e^{-j\frac{8\pi}{5}\frac{3}{2}} - 3\,\frac{\sin\left(4\cdot\frac{2\pi}{5}\,\frac{1}{2}\right)}{k\pi} e^{-j\frac{8\pi}{5}\frac{9}{2}}$$

$$= \frac{\sin\left(\frac{12\pi}{5}\right)}{4\pi} e^{-j\frac{12\pi}{5}} - 3\,\frac{\sin\left(\frac{4\pi}{5}\right)}{4\pi} e^{-j\frac{32\pi}{5}}$$

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This sheet is for Question 1.

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3. Using the differentiation property

$\because \quad y(t) = \dfrac{d\,x(t)}{dt}$

$b_k = jk\omega_0 a_k$

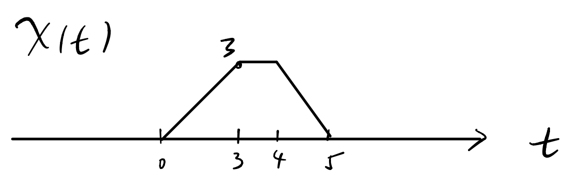
$a_k = \dfrac{1}{jk\omega_0} b_k$

$\therefore \quad a_4 = \dfrac{1}{j4\frac{2\pi}{5}} b_4$

$= \dfrac{5}{j8\pi}\left(\dfrac{1}{j2\pi} - \dfrac{1}{j8\pi} e^{-j\frac{24\pi}{5}} - \dfrac{3}{j8\pi} e^{-j\frac{32\pi}{5}}\right)$

$= -\dfrac{5}{16\pi^2} + \dfrac{5}{64\pi^2} e^{-j\frac{24\pi}{5}} + \dfrac{15}{64\pi^2} e^{-j\frac{32\pi}{5}}$

$a_0 = \dfrac{1}{T_0}\int_{T_0} x(t) e^{j0\omega_0 t} dt = \dfrac{1}{5}\left(\dfrac{9}{2} + 3 + \dfrac{3}{2}\right) = \dfrac{9}{5}$



periodic $T_0 = 5$

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Question 2: [14%, Work-out question] Consider a DT signal $x[n]$:

$$x[n] = \begin{cases} n + j & \text{if } 0 \le n \le 3 \\ -2j & \text{if } 4 \le n \le 5 \\ 0 & \text{if } 6 \le n \le 7 \\ \text{periodic with period } N = 8 \end{cases} \quad (3)$$

Denote its DTFS by $(a_k, \frac{2\pi}{8})$ where a_k is the DTFS coefficient.

1. [7%] Find the value of $\sum_{k=-6}^{-3} a_k + \sum_{k=6}^{9} a_k$
2. [7%] Find the value of $\sum_{k=0}^{7} |a_k|^2$

Answer:

1. $\because N = 8$

$$\sum_{k=-6}^{-3} a_k + \sum_{k=6}^{9} a_k$$

$$= \sum_{k=0}^{7} a_k$$

By the synthesis equation

$$x[0] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{8}0} = \sum_{k=\langle N \rangle} a_k$$

$$\therefore \sum_{k=-6}^{-3} a_k + \sum_{k=6}^{9} a_k = x[0] = j$$

2. By Parseval's relation

$$\sum_{k=0}^{7} |a_k|^2 = \frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2$$

$$= \frac{1}{8} \left(\sum_{n=0}^{3} |n+j|^2 + \sum_{n=4}^{5} |-2j|^2 \right)$$

$$= \frac{1}{8} (1 + 2 + 5 + 10 + 8)$$

$$= \frac{13}{4}.$$

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Question 3: [16%, Work-out question]

Consider two DT signals $x[n] = \cos(\frac{3\pi}{5}n)$ and $y[n] = \sin(\frac{11\pi}{5}n)$. Define $z[n] = x[n] \cdot y[n]$.

Denote the DTFS of $x[n]$ by $(a_k, \frac{2\pi}{N_x})$, denote the DTFS of $y[n]$ by $(b_k, \frac{2\pi}{N_y})$, and denote the DTFS of $z[n]$ by $(c_k, \frac{2\pi}{N_z})$.

1. [5%] Find the DTFS coefficients a_k of $x[n]$.
2. [5%] Find the DTFS coefficients b_k of $y[n]$.
3. [6%] Find the values of c_1 and c_4, respectively.

   Hint: If you do not know the answers to Q3.1 and Q3.2, you may assume $a_k = \sin(0.2\pi k)$ and $b_k = \cos(0.2\pi k)$. You will receive 5 points for Q3.3 if your answers are correct (using the given a_k and b_k values).

Answer: 1) $x[n] = \cos(\frac{3\pi}{5}n) = \frac{1}{2}(e^{-j\frac{3\pi}{5}n} + e^{j\frac{3\pi}{5}n})$

$\frac{3\pi}{5}N = 2\pi m \quad m \in \mathbb{Z} \Rightarrow N = 10$

$a_{-3} = a_3 = \frac{1}{2}, \quad a_k = 0$ for $k \in [-3, 6]$ & $k \neq -3, 3$.

2) $b[n] = \sin(\frac{11\pi}{5}n) = \frac{1}{2j}(e^{j\frac{11\pi}{5}n} - e^{-j\frac{11\pi}{5}n})$

$\frac{11\pi}{5}N = 2\pi m \quad m \in \mathbb{Z} \Rightarrow N = 10.$

$b_{11} = b_1 = \frac{1}{2j} \quad b_{-11} = b_9 = -\frac{1}{2j}$

$b_k = 0$ for $k \in [0, 9]$ & $k \neq 1, 9$.

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This sheet is for Question 3.

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3)

$$C_1 = \sum_{k=\langle 10\rangle} a_k b_{1-k}$$

$$= \underset{k=3}{a_3 b_{-2}} + \underset{k=7}{a_7 b_{-6}}$$

$$= a_3 b_8 + a_7 b_4$$

$$= 0 + 0 = 0$$

$$C_4 = \sum_{k=\langle 10\rangle} a_k b_{4-k}$$

$$= a_3 b_1 + a_7 b_{-3}$$

$$= \frac{1}{2}\,\frac{1}{2j} + 0$$

$$= \frac{1}{4j}$$

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Question 4: [18%, Work-out question] Consider a CT-LTI system with impulse response

$$h(t) = \sqrt{2}e^{-|t|} \tag{4}$$

We use the signal $x(t) = \sum_{k=1}^{10} \cos(k^2(0.2\pi t))$ as the input to the above LTI system and denote the corresponding output by $y(t)$. Find the expression of $y(t)$.

Hint 1: If you do not know how to solve this question, you can assume $x(t) = \cos(3\pi t)$ and use this simpler $x(t)$ to find the output $y(t)$. **You will receive 15 points out of 18 points** if your answer is correct.

Hint 2: Your answer could be something of the following form:
E.g., $y(t) = \sum_{k=3}^{20} \frac{1}{1+jk} e^{k\cdot 3\cdot \pi t - k}$

Answer:

$$H(j\omega) = \int_{t=-\infty}^{+\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \int_{t=-\infty}^{+\infty} \sqrt{2}\, e^{-|t|}\, e^{-j\omega t}\, dt$$

$$= \sqrt{2}\int_{t=0}^{+\infty} e^{-(1+j\omega)t}\, dt + \sqrt{2}\int_{t=-\infty}^{0} e^{-(-1+j\omega)t}\, dt$$

$$= \sqrt{2}\, \frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\Bigg|_{t=0}^{+\infty} + \sqrt{2}\, \frac{e^{-(-1+j\omega)t}}{-(-1+j\omega)}\Bigg|_{t=-\infty}^{0}$$

$$= \sqrt{2}\, \frac{1}{1+j\omega} + \sqrt{2}\, \frac{1}{1-j\omega}$$

$$= \frac{2\sqrt{2}}{1+\omega^2}$$

$$e^{j\omega_0 t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(j\omega_0)e^{j\omega_0 t}$$

By lineary of $h(t)$:

$$y(t) = \sum_{k=1}^{10} \frac{2\sqrt{2}}{1+\left(\frac{k^2}{5}\pi\right)^2} \cos\left(\frac{k^2}{5}\pi t\right)$$

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Question 5: [18%, Work-out question]

Consider the following CT signal $x(t)$:

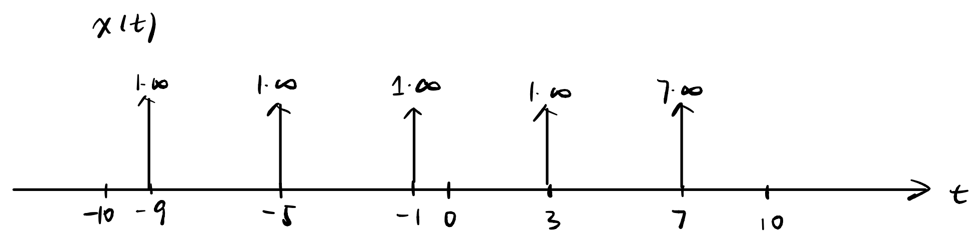
$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - 4k + 1) \quad (5)$$

1. [2%] Plot $x(t)$ for the range of $-10 \le t \le 10$.
2. [11%] Find the expression of $X(j\omega)$.

   Hint: If you don't know how to solve Q5.2, you can find the CTFS of $x(t)$ instead. You will receive 10 points if your answer is correct.
3. [5%] Plot $X(j\omega)$ for the range of $-0.6\pi < \omega < 0.6\pi$.

Answer:

1)



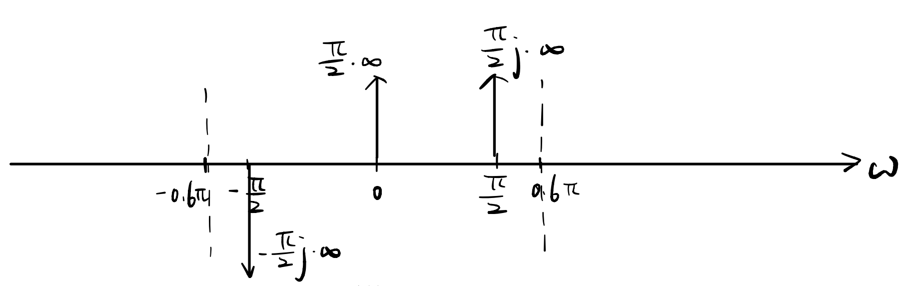
2) Let $x_0(t) = \sum_{k=-\infty}^{+\infty} \delta(t-4k)$

$x(t) = x_0(t+1)$

By the time-shifting property

$$X(j\omega) = e^{+j\omega} X_0(j\omega) = \frac{\pi}{2} e^{j\omega} \sum_{k=-\infty}^{+\infty} \delta(\omega - \frac{\pi}{2}k)$$

3)



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Question 6: [18%, Work-out question]
Consider the following two CT signals:

$$x(t) = \frac{\sin(2t)}{\pi t} \tag{6}$$

$$y(t) = \frac{\sin(10t)}{\pi t} \tag{7}$$

Define $z(t) = ((x(t) \cdot \sin(5t)) \cdot \cos(5t)) * y(t)$.

1. [18%] Plot $Z(j\omega)$ for the range of $-15 \le \omega \le 15$.

   Hint: You may want to find $X(j\omega)$ first and then gradually find $Z(j\omega)$. You will receive partial credit if you do it step-by-step.

Answer: $z(t) = \left(x(t) \cdot (\sin(5t) \cdot \cos(5t))\right) * y(t)$

$= \left(x(t) \cdot \frac{1}{2}\sin(10t)\right) * y(t)$

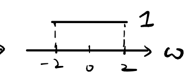
$= \frac{1}{4} x(t) \cdot \left(\frac{1}{j}e^{j10t} - \frac{1}{j}e^{-j10t}\right) * y(t)$

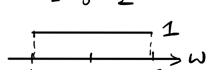
Using the frequency shifting property and convolution property

$Z(j\omega) = \frac{1}{4j}\left(X(j(\omega-10)) - X(j(\omega+10))\right) Y(j\omega)$

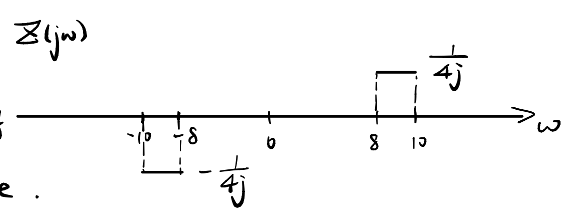
$= \frac{1}{4j} X(j(\omega-10)) Y(j\omega) - \frac{1}{4j} X(j(\omega+10)) Y(j\omega)$

Using the transform pairs:

$x(t) = \frac{\sin(2t)}{\pi t} \longrightarrow$ 

$y(t) = \frac{\sin(10t)}{\pi t} \longrightarrow$ 

$$Z(j\omega) = \begin{cases} \frac{1}{4j} & 8 < \omega < 10 \\ -\frac{1}{4j} & -10 < \omega < -8 \\ 0 & \text{otherwise.} \end{cases}$$



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Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2. \quad (3.[illegible])$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—