Midterm #2 of ECE 301-004, (CRN: 13890)
8–9pm, Tuesday, March 1, 2022, FRNY G140.

1. **<u>Do not write answers on the back of pages!</u>**
2. **After the exam ended, you will have 5 additional minutes to write down your name and Purdue ID on each of the pages.**
3. **If you need additional sheets to write down your answers, please let the instructor/TA know. We will hand out additional answer sheets then.**
4. Enter your student ID number, and signature in the space provided on this page.
5. This is a closed book exam.
6. This exam contains multiple-choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.
7. The instructor/TA will hand out loose sheets of paper for the rough work.
8. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

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Question 1: [16%, Work-out question]
Consider the following signal

$$x(t) = \begin{cases} \cos(t+4) + j\sin(t+4) & \text{if } 1 \leq t \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Also consider a CT-LTI system with impulse response being

$$h(t) = \begin{cases} e^{2t} & \text{if } t > 3 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

1. [2%] What is the definition of an *impulse response*?
2. [14%] Find the output signal $y(t)$ when we feed $x(t)$ to the above CT-LTI system as input.

   Hint: Your answer can be of the following form $\frac{e^{j\pi t+4}}{3} - \frac{e^{-j\pi+4t}}{5} + \frac{e^{2jt-2}}{3}$. There is no need to further simplify it.

Answer:

1. An impulse response is the output of a system when the input is an unit impulse.

2.
$$x(\tau) = \begin{cases} e^{j(\tau+4)} & 1 \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

$$h(t-\tau) = \begin{cases} e^{2(t-\tau)} & t-\tau > 3 \Rightarrow \tau < t-3 \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = \int_{\tau=-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

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This sheet is for Question 1.

$$y(t) = \int_{\tau=-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Case 1 $t-3 < 1 \Rightarrow t < 4$

$$y(t) = 0$$

Case 2 $t-3 \geq 1 \Rightarrow t \geq 4$

$$y(t) = \int_{\tau=1}^{t-3} e^{j(\tau+4)}\, e^{2(t-\tau)}\, d\tau$$

$$= \int_{\tau=1}^{t-3} e^{2t+4j}\, e^{(-2+j)\tau}\, d\tau$$

$$= e^{2t+4j} \left. \frac{e^{(-2+j)\tau}}{-2+j} \right|_{\tau=1}^{t-3}$$

$$= e^{2t+4j} \frac{e^{(-2+j)(t-3)} - e^{(-2+j)}}{-2+j}$$

$$= \frac{e^{2t+4j} \cdot e^{(-2+j)} \left(e^{t-3} - 1\right)}{-2+j}$$

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Question 2: [20%, Work-out question]

1. [14%] Consider a DT-LTI system and we know that if the input is

$$x[n] = \begin{cases} 1 & \text{if } 0 \leq n \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

   then the output is

$$y[n] = \begin{cases} 1 & \text{if } -1 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

   Plot the output $y_{\text{new}}[n]$ for the range of $-10 \leq n \leq 10$ when the input is

$$x_{\text{new}}[n] = \begin{cases} 1 & \text{if } 0 \leq n \leq 2 \\ 2 & \text{if } 3 \leq n \leq 5 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

   Hint 1: If you do not know how to solve this question, solve the following question instead. We know that $\delta[n]$ can be expressed as $\delta[n] = \sum_{k=-\infty}^{\infty} a_k \cdot U[n-k]$ where $U[n]$ is the *unit step signal*. Find the coefficient a_k for all k. You will receive 5 points if your answer for the alternative Q2.1 is correct.

2. [6%] Consider a CT-LTI system with impulse response $h(t) = (\cos(t) + j\sin(t))U(t)$ where $U(t)$ is the unit step signal. Is such a system stable?

   This is a work-out question and you need to carefully justify your answer. An answer without justification would not receive any point.

   Hint 2: If you do not know the answer to this question, please write down the definition of "stable systems". You will receive 3 points if your answer is correct.

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This sheet is for Question 2.

Answer :

$$x[n] = \begin{cases} 1 & \text{if } 0 \le n \\ 0 & \text{otherwise} \end{cases}$$

1. From step response to impulse response:

$$y[n] = \begin{cases} 1 & \text{if } -1 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$x_{\text{new}}[n] = \begin{cases} 1 & \text{if } 0 \le n \le 2 \\ 2 & \text{if } 3 \le n \le 5 \\ 0 & \text{otherwise} \end{cases}$$

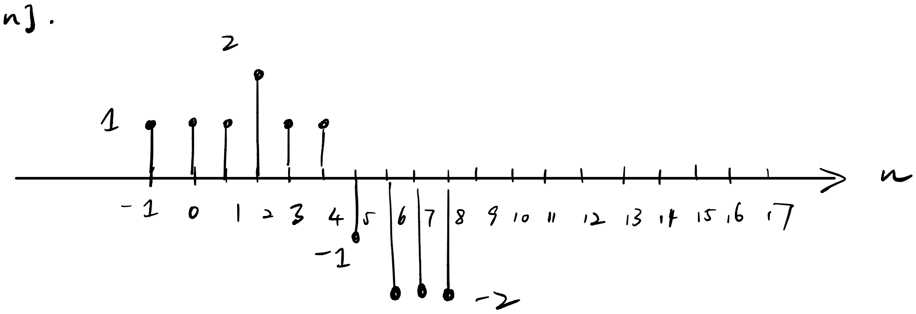
$x[n] \longrightarrow y[n]$

$x[n-1] \longrightarrow y[n-1]$ ($\because$ time invariant)

$\delta[n] = x[n] - x[n-1] \Rightarrow h[n] = y[n] - y[n-1]$

$$h[n] = \begin{cases} 1 & \text{when } n = -1, \\ -1 & \text{when } n = 3, \\ 0 & \text{otherwise.} \end{cases}$$

$y[n]$.



2. $h(t) = e^{jt} U(t)$

$$h(t) = (\cos(t) + j\sin(t))U(t)$$

$$\int_{t=-\infty}^{+\infty} |h(t)|\, dt = \int_{t=0}^{+\infty} 1 \, dt = \infty$$

It is not stable.

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Question 3: [15%, Work-out question]

Consider two CT-LTI systems. The first CT-LTI system has its impulse response being

$$h_1(t) = U(t) - U(t-1) \tag{6}$$

and the second CT-LTI system has its impulse response being

$$h_2(t) = e^{-t}U(t) \tag{7}$$

Consider a new signal $x(t) = e^{-j2.5t}$. We pass $x(t)$ through the first system and denote its output by $y(t)$. We then use the output of the first system $y(t)$ as the input to the second system and denote the final output by $z(t)$. That is, $z(t)$ is the final output if we use $x(t)$ as the input to the *sequentially concatenated system.*

Question: Find the expression of $z(t)$.

Hint 1: Your answer can be of the following form $\frac{e^{j\pi t+4}}{3} \cdot \frac{e^{-j\pi+4}}{5} \cdot \frac{e^{2j-2}}{3}$. There is no need to further simplify it.

Hint 2: If you do not know the answer to this question, please find the period of $y(t)$. You will receive 6 points if your answer is correct.

Answer:

$$H_1(e^{j\omega}) = \int_{t=-\infty}^{+\infty} h_1(t) e^{-j\omega t} dt$$

$$= \int_{t=0}^{1} e^{-j\omega t} dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{t=0}^{1}$$

$$= \frac{e^{-j\omega} - 1}{-j\omega}$$

$$H_2(e^{j\omega}) = \int_{t=0}^{+\infty} e^{-t} e^{-j\omega t} dt$$

$$= \left. \frac{e^{-(1+j\omega)t}}{-(1+j\omega)} \right|_{t=0}^{+\infty} = \frac{1}{1+j\omega}.$$

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This sheet is for Question 3.

$$H(j\omega) = H_1(j\omega) H_2(j\omega)$$

$$= \frac{e^{-j\omega} - 1}{-j\omega} \cdot \frac{1}{1 + j\omega}$$

$$z(t) = \frac{e^{j2.5} - 1}{-j2.5} \cdot \frac{1}{1 - j2.5} e^{-j2.5t}$$

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Question 4: [17%, Work-out question] Consider a periodic CT signals

$$x(t) = \begin{cases} 2 & \text{if } 0 \leq t < 1 \\ -1 & \text{if } 1 \leq t < 2 \\ 2 & \text{if } 2 \leq t < 6 \\ \text{periodic with period } T = 6 \end{cases} \quad (8)$$

We denote the Fourier series of $x(t)$ by (a_k, ω_0).

1. [5%] Find the value of a_0.
2. [12%] Find the general expression of a_k for all $k \neq 0$.

   Hint: If may be easier to do the following three steps: (1) First express $x(t)$ as some kind of transformation of a different signal $y(t)$; (2) find the CTFS of $y(t)$; and (3) finally find the CTFS of $x(t)$.

Answer: 1. $a_0 = \frac{1}{T}\int_0^6 x(t) e^{-j 0 \cdot \omega_0 t} dt$

$= \frac{1}{6}(1 \times 2 - 1 + 2 \times 4)$

$= 3/2$

2. $a_k = \frac{1}{T}\int_{t=0}^{6} x(t) e^{-jk\omega_0 t} dt$

$= \frac{1}{6}\int_{t=0}^{1} 2 e^{-jk\frac{\pi}{3}t} dt + \frac{1}{6}\int_{t=1}^{2} (-1) e^{-jk\frac{\pi}{3}t} dt$

$+ \frac{1}{6}\int_{t=2}^{6} 2 e^{-jk\frac{\pi}{3}t} dt$

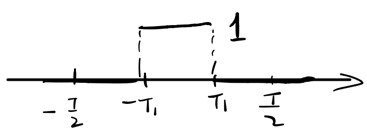
$= \frac{1}{6}\left(2 \frac{e^{-jk\frac{\pi}{3}t}}{-jk\frac{\pi}{3}}\Big|_{t=0}^{1} - 1 \cdot \frac{e^{-jk\frac{\pi}{3}t}}{-jk\frac{\pi}{3}}\Big|_{t=1}^{2} + 2 \frac{e^{-jk\frac{\pi}{3}t}}{-jk\frac{\pi}{3}}\Big|_{t=2}^{6}\right)$

$$= \frac{1}{6}\left(\frac{2e^{-jk\frac{\pi}{3}} - 2}{-jk\frac{\pi}{3}} - \frac{e^{-jk\frac{\pi}{3}\cdot 2} - e^{-jk\frac{\pi}{3}}}{-jk\frac{\pi}{3}} + 2\,\frac{e^{-jk\frac{\pi}{3}6} - e^{-jk\frac{\pi}{3}2}}{-jk\frac{\pi}{3}} \right)$$

$$= -\frac{1}{jk2\pi}\left(3e^{-jk\frac{\pi}{3}} - 3e^{-jk\frac{2}{3}\pi} \right)$$

Method 2:

Use the result that



$$\overset{F}{\longleftrightarrow} \quad a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \qquad k \neq 0.$$

$$x(t) \overset{F}{\longleftrightarrow} a_k \qquad k \neq 0$$

$$a_k = 2\,\frac{\sin(k\frac{2\pi}{6}1)}{k\pi}e^{-jk\frac{2\pi}{6}\cdot\frac{1}{2}} - \frac{\sin(k\frac{2\pi}{6}1)}{k\pi}e^{-jk\frac{2\pi}{6}\frac{3}{2}}$$

$$+ 2\,\frac{\sin(k\frac{2\pi}{6}2)}{k\pi}e^{-jk\frac{2\pi}{6}4}$$

$$= -\frac{1}{jk2\pi}\left(3e^{-jk\frac{\pi}{3}} - 3e^{-jk\frac{2}{3}\pi} \right)$$

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Question 5: [12%, Work-out question]

Consider a CT signal $x(t)$ of period $T = 8$. Suppose we know that the corresponding Fourier series coefficients are $a_3 = 1 + j$ and $a_{-3} = 1 - j$ and all other $a_k = 0$.

1. [12%] Find the expression of the *real-part* of $x(t)$.

Answer:

$$x(t) = a_3 \cdot e^{j3\frac{2\pi}{8}t} + a_{-3}e^{-j3\frac{2\pi}{8}t}$$

$$= (1+j)e^{j\frac{3}{4}\pi t} + (1-j)e^{-j\frac{3}{4}\pi t}$$

$$= 2\cos\left(\frac{3}{4}\pi t\right) - 2\sin\left(\frac{3}{4}\pi t\right)$$

$$x_{real}(t) = x(t)$$

$$= 2\left(\cos\left(\frac{3}{4}\pi t\right) - \sin\left(\frac{3}{4}\pi t\right)\right)$$

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Question 6: [20%, Multiple Choices]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \int_{s=2t-1}^{2t+1} |x(0.5s)| + x(0.5s)ds \tag{9}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = \sum_{k=0}^{|x_2[n]|} (n-k) \tag{10}$$

Answer the following questions

1. [4%] Is System 1 memoryless? Is System 2 memoryless?
2. [4%] Is System 1 causal? Is System 2 causal?
3. [4%] Is System 1 stable? Is System 2 stable?
4. [4%] Is System 1 linear? Is System 2 linear?
5. [4%] Is System 1 time-invariant? Is System 2 time-invariant?

System 1:

not - memoryless

not - Causal

Stable

not - linear

time invariant.

System 2

memoryless

Causal

not - stable

not - linear

not - time invariant.

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This sheet is for Question 6.

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} \quad [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} \quad [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t - 1) - 1/2.$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	Na_kb_k
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$