PURDUE

COURSENAME/SECTIONNUMBER
EXAM TITLE

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Tips for making sure GradeScope can read your exam:

1. Make sure your name and PUID are clearly written at the top of every page, including any additional blank pages you use.
2. Write only on the front of the exam pages.
3. Add any additional pages used to the back of the exam before turning it in.
4. Ensure that all pages are facing the same direction.
5. Answer all questions in the area designated for that answer. Do not run over into the next question space.

Final Exam of ECE 301-004, (CRN: 13890)
3:30–5:30pm, Thursday, May 5, 2022, PHYS 112.

1. **Do not write answers on the back of pages!**
2. **After the exam ended, you will have 5 additional minutes to write down your name and Purdue ID on each of the pages.**
3. **If you need additional sheets to write down your answers, please let the instructor/TA know. We will hand out additional answer sheets then.**
4. Enter your student ID number, and signature in the space provided on this page.
5. This is a closed book exam.
6. This exam contains multiple-choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have two hours to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.
7. If needed and requested by students, the instructor/TA will hand out loose sheets of paper for the rough work.
8. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

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Question 1: [23%, Work-out question]

1. [1%] What does the acronym FDM stand for? It stands for asynchronous frequency division multiplexing.

Prof. Wang wanted to transmit an AM-SSB signal. To that end, he wrote the following MATLAB code.

```
% Initialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read two different .wav files
[x1, f_sample, N]=audioread('x1.wav');
x1=x1';
[x2, f_sample, N]=audioread('x2.wav');
x2=x2';

% Step 0: Initialize several parameters
W_1=3000*pi;
W_2=1500*pi;
W_3=????;
W_4=????;
W_5=????;
W_6=????;
W_7=8000*pi;


% Step 1: Make the signals band-limited.
h=1/(pi*t).*(sin(W_1*t));
x1_new=ece301conv(x1, h);
x2_new=ece301conv(x2, h);

% Step 2: Multiply x1_new and x2_new with a sinusoidal wave.
x1_h=x1_new.*sin(W_2*t);
x2_h=x2_new.*sin(W_3*t);

% Step 3: Keep one of the two side bands
h_one=1/(pi*t).*(sin(W_4*t).*(2*cos(W_5*t)));
h_two=1/(pi*t).*(sin(W_6*t)-sin(W_7*t));
x1_sb=ece301conv(x1_h, h_one);
```

```
x2_sb=ece301conv(x2_h, h_two);

% Step 4: Create the transmitted signal
y=x1_sb+x2_sb;
audiowrite('y.wav', y, f_sample);
```

2. [1%] What is the carrier frequency (Hz) of the signal x1_new? 750 Hz.

3. [4%] Our goal is to transmit either the lower-side band (LSB) or the upper side band (USB). However, it turns out that for the x1 signal, only one of the two options (LSB and USB) is possible. Explain in details which option (LSB or USB) is possible for the x1 signal. USB, because the LSB spectrums are overlapped.

   Hint: You need to carefully justify your answer. An answer without justification will receive zero point.

4. [3%] Continue from the previous sub-question. What would be the right values of W_4 and W_5? $W_4 = 1500\pi$, $W_5 = 3000\pi$

5. [3%] If I would like to transmit the upper side band of the x2 signal. What would be my choice of W_3 and W_6 values? When answering this sub-question, please always assume the value of W_6 is no less than $W_7 = 8000\pi$. I.e., $W_6 \geq 8000\pi$.

   $W_3 = 8000\pi$, $W_6 = 11000\pi$,

6. [2%] If I would like to transmit the lower side band of the x2 signal. What would be my choice of W_3 and W_6 values? When answering this sub-question, please always assume the value of W_6 is no less than $W_7 = 8000\pi$. I.e., $W_6 \geq 8000\pi$.

   $W_3 = 11000\pi$, $W_6 = 11000\pi$

Prof. Wang decided to use **the upper-side-band transmission** for both the x1 and the x2 signals, and used the code in the previous page to generate the "y.wav" file.

A student tried to demodulate the output waveform "y.wav" by the following code.

```
% Initialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read the .wav files
[y, f_sample, N]=audioread('y.wav');
y=y';

% Initialize several parameters


W_8=????;
W_9=6000*pi;
W_10=1500*pi;
W_11=1500*pi;
W_12=7000*pi;
W_13=????;





% Create a new low-pass filter.
h_M=1/(pi*t).*(sin(W_8*t));


% We construct new BPFs
h_three=1/(pi*t).*(sin(W_9*t)-sin(W_10*t));
h_four=1/(pi*t).*(sin(W_11*pi*t).*(2*cos(W_12*t)));

% demodulate signal 1
y11=ece301conv(y, h_three);
y1=y11.*sin(1500*pi*t);
x1_hat=4*ece301conv(y1,h_M);

sound(x1_hat,f_sample)
```

```
% demodulate signal 2
y21=ece301conv(y, h_four);
y2=y21.*sin(W_13*t);
x2_hat=4*ece301conv(y2,h_M);

sound(x2_hat,f_sample)
```

7. [3%] Continue from the previous questions. What should the values of W_8 and W_13 be in the MATLAB code? $W_8 = 3000\pi$, $W_{13} = 8000\pi$.

8. [5%] It turns out that the above MATLAB code is not written correctly and part of the end results do not sound right. Answer the following questions

   Yes (a) Is signal x1_new correctly/perfectly demodulated? If yes, then go to subquestion (d). If no, then continue answering the following sub-questions.

   (b) Use 2 to 3 sentences to answer (i) what kind of problem does x1_new have, i.e., how does the problem impact the sound quality of "sound(x1_hat,f_sample)"?

   (c) How can the MATLAB code be corrected so that the playback/demodulation can be successful?

   No. (d) Is signal x2_new correctly/perfectly demodulated? If yes, then your answer to Q1.8 is complete. If no, then continue answering the following sub-questions.

   (e) Use 2 to 3 sentences to answer (i) what kind of problem does x2_new have, i.e., how does the problem impact the sound quality of "sound(x2_hat,f_sample)"?

   (f) How can the MATLAB code be corrected so that the playback/demodulation can be successful?

   Change W_{11} and W_{12} ; $W_{11} = 1500\pi$ $W_{12} = 9500\pi$.

   Hint: If you do not know the answers of Q1.3 to Q1.8, please simply draw the AMSSB modulation (using lower side band) and demodulation diagrams and mark carefully all the parameter values. You will receive 12 points for Q1.3 to Q1.8 if your system diagrams are correct and all parameter values are marked correctly.

8 (e)

X2_new becomes quieter, because the high frequency components are discarded.

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Question 2: [14%, Work-out question]

Consider a continuous time signal:

$$x(t) = U(t+5) - U(t-5) \tag{1}$$

where $U(t)$ is the unit-step signal.

1. [1%] Plot $x(t)$ for the range of $-10 \le t \le 10$.
2. [3%] Plot $X(j\omega)$, the CTFT of $x(t)$, for the range of $-\pi \le \omega \le \pi$.

We construct another signal $y(t) = x(t) \cdot \cos(\pi t)$.

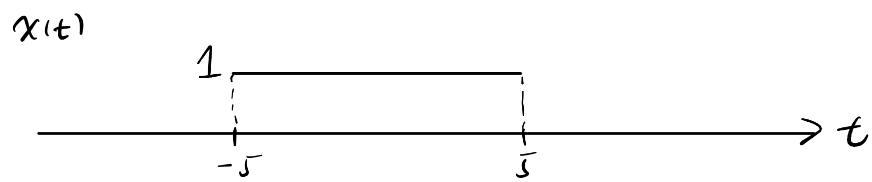
3. [5%] Plot $Y(j\omega)$, the CTFT of $y(t)$, for the range of $-1.4\pi \le \omega \le 1.4\pi$.

   Hint 1: If you don't know how to answer this question, you can write down the relationship between $X(j\omega)$ and $Y(j\omega)$. You will receive 2.5 points if your answer is correct.

Suppose we perform amplitude modulation to convert an acoustic $w(t) = \sin(\pi t)$ to a new signal $z(t) = w(t) \cdot \sin(4000\pi t)$.

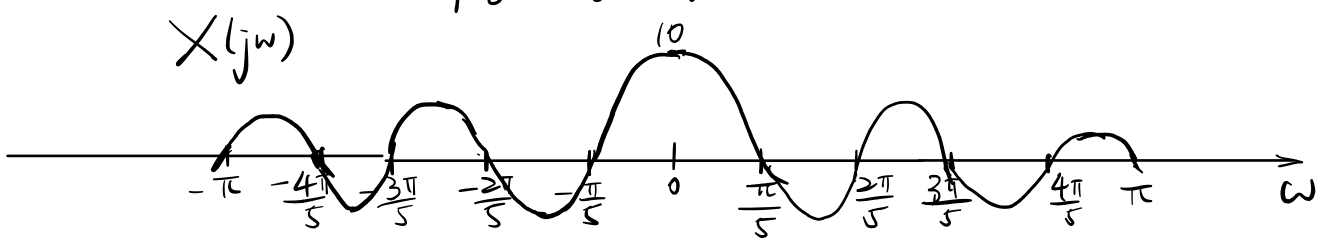
4. [5%] If we demodulate the signal $z(t)$ by the *asynchronous demodulation*. Denote the final output by $\hat{w}(t)$. Plot $\hat{w}(t)$ for the range of $-4 \le t \le 4$.

Answer: 1.



2. According to the transform pair

$$x(t) = \begin{cases} 1 & |t| \le 5 \\ 0 & \text{otherwise} \end{cases} \Longleftrightarrow X(jw) = \frac{2\sin(5w)}{w}.$$



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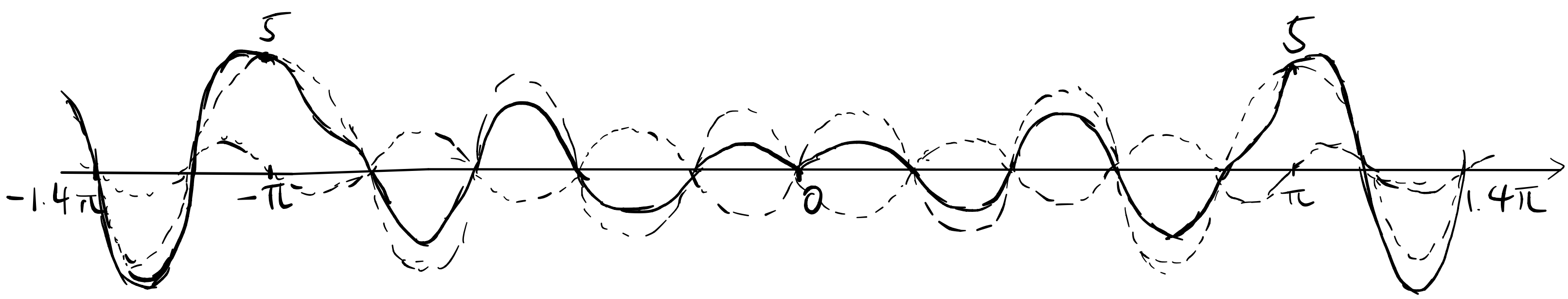
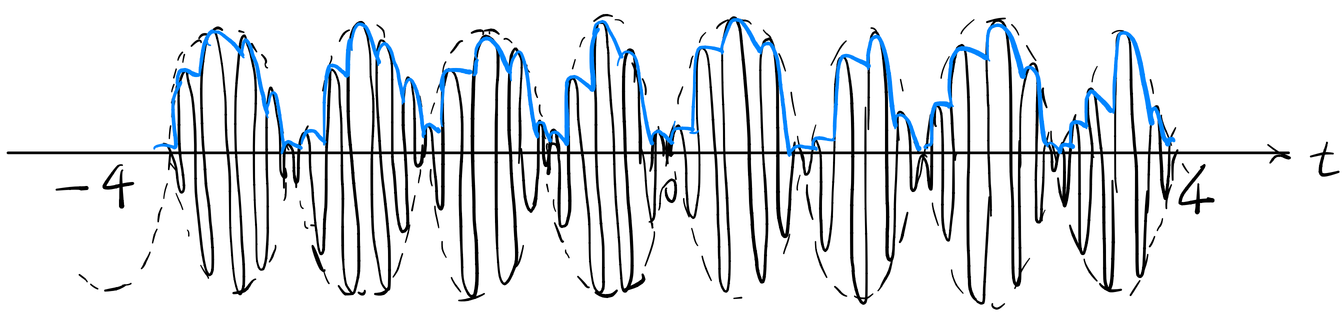
This sheet is for Question 2.

Using the multiplication property

3.



4. $\hat{w}(t)$



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Question 3: [12%, Work-out question]

1. [4%] Consider a continuous time signal

$$x(t) = \begin{cases} 2t+2 & \text{if } -1 \le t < 0 \\ 2 & \text{if } 0 \le t < 0.5 \\ 3-2t & \text{if } 0.5 \le t < 1.5 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

   We sample $x(t)$ with the sampling frequency 2Hz and denote the sampled values by $x[n]$. Plot $x[n]$ for the range of $-5 \le n \le 5$.

2. [3%] We also perform 2Hz *Impulse Train Sampling* (ITS) on $x(t)$ and the resulting signal is $x_p(t)$. Plot $x_p(t)$ for the range of $-5 \le t \le 5$.

3. [5%] Define another signal

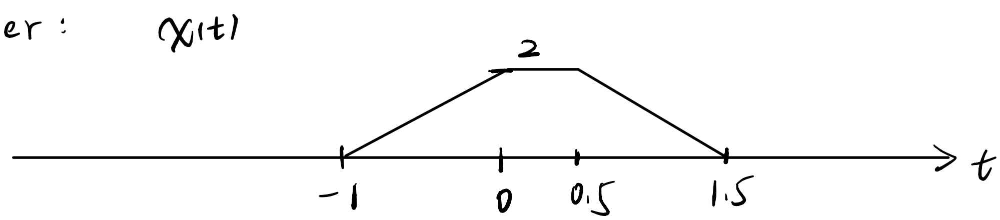
$$h(t) = \begin{cases} 1 & \text{if } |t| < 0.25 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

   Denote $y(t) = x_p(t) * h(t)$. Plot $y(t)$ for the range of $-5 \le t \le 5$.

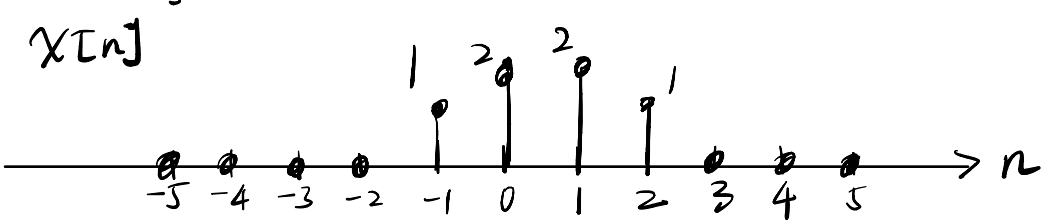
   Hint: If you do not know the answer to this subquestion, you can assume $y(t) = \sin(\pi t) * \delta(t-1.5)$ and plot $y(t)$ for the range of $-5 \le t \le 5$. You will receive 2 points if your answer is correct.

Answer:

1.



$T_s = \frac{1}{f_s} = 0.5s$



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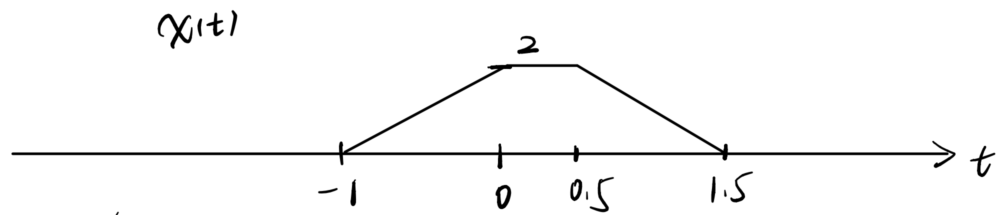
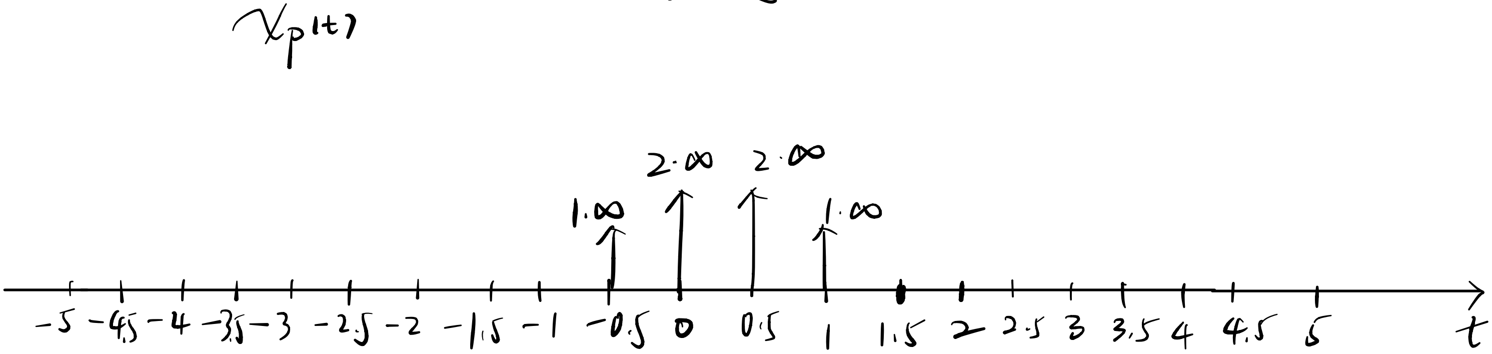
2.





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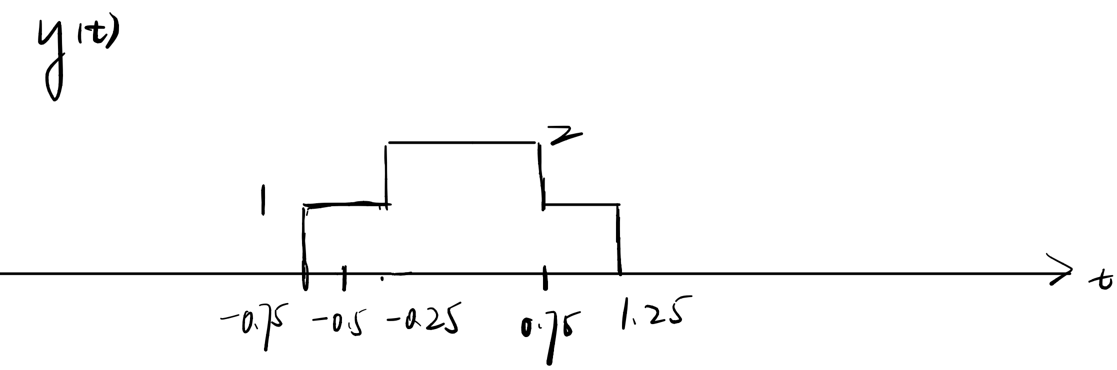

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Question 4: [13%, Work-out question]

1. [5%] Consider a continuous time signal $x(t) = \sin(\pi t)$. We sample $x(t)$ via *impulse train sampling* with sampling period 0.2. Denote the final *impulse-train-sampled* signal by $x_p(t)$. Plot $X_p(j\omega)$, the CTFT of $x_p(t)$, for the range of $-15\pi < \omega < 15\pi$;

2. [3%] We pass $x_p(t)$ through an ideal band-pass filter of cutoff frequencies $W_L = 8\pi$ and $W_H = 10\pi$ and denote the output by $y(t)$. Plot $Y(j\omega)$, the CTFT of $y(t)$, for the range of $-15\pi < \omega < 15\pi$;

   Hint 1: if you do not know the answer to Q4.1 and Q4.2, you can simply write down the impulse response of the ideal BPF with $W_L = 8\pi$ and $W_H = 12\pi$. You will receive 1.5 points of your answer is correct.

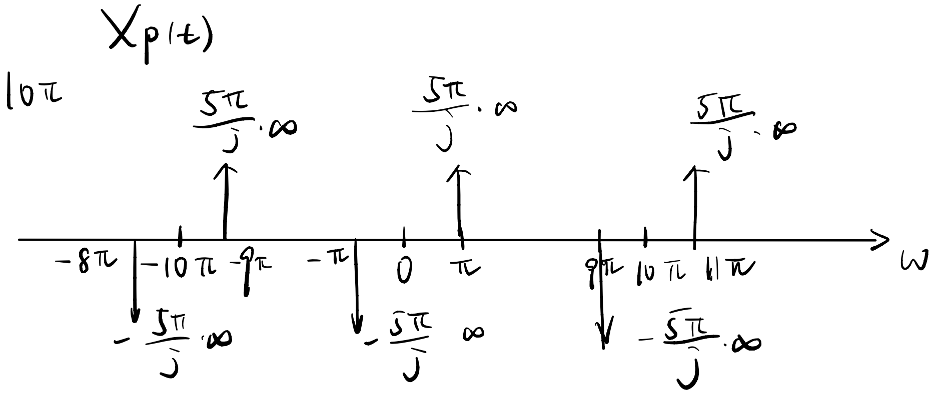
3. [5%] Find the expression of $y(t)$.

   Hint 2: You do not need to plot $y(t)$. Just writing down the mathematical expression of $y(t)$ would suffice.

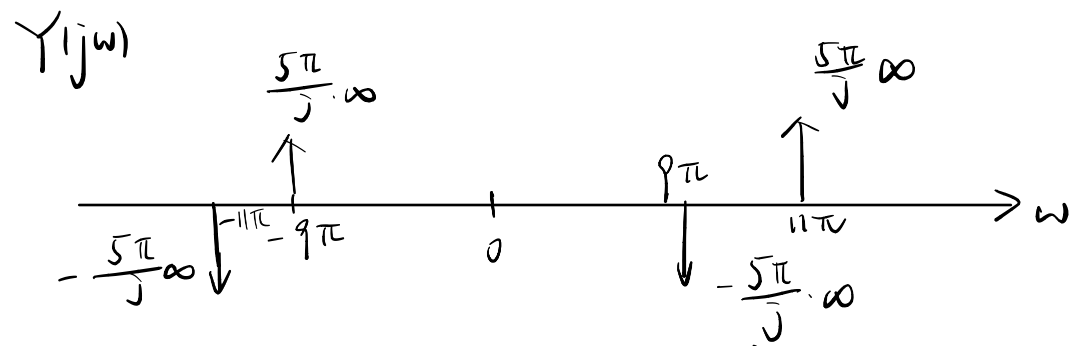
   Hint 3: This process is sometimes termed the *Amplitude Modulation Via Impulse Train Sampling.* This "name" should be helpful when you are answering Q4.3.

Answer:

1. $\omega_s = \frac{2\pi}{0.2} = 10\pi$



2.



3. $y(t) = 5\sin(11\pi t) - 5\sin(9\pi t)$

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Question 5: [8%, Work-out question]

Consider the following discrete time signals

$$x[n] = \begin{cases} n \cdot e^{j0.25\pi n} & \text{if } 1 \le n \le 4 \\ 1+j & \text{if } 5 \le n \le 20 \\ \text{periodic with period } N = 20 & \end{cases} \quad (4)$$

Denote the DTFS coefficients of $x[n]$ by a_k.

1. [4%] Find the value of $\sum_{k=0}^{39} a_k$.
2. [4%] Find the value of $\sum_{k=10}^{29} |a_k|^2$.

Answer:

1. Using the synthesis equation

$$x[0] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}\cdot 0}$$

$$= \sum_{k=\langle N \rangle} a_k$$

$$\therefore \sum_{k=0}^{39} a_k = 2x[0] = 2(1+j) = 2+2j.$$

2. Using the Parseval's relation

$$\sum_{k=10}^{29} |a_k|^2 = \sum_{k=1}^{20} |a_k|^2 = \frac{1}{20} \sum_{n=\langle N \rangle} |x[n]|^2$$

$$= \frac{1}{20}\left(\sum_{n=1}^{4} |n \cdot e^{j\frac{\pi}{4}n}|^2 + \sum_{n=5}^{20} |1+j|^2 \right)$$

$$= \frac{1}{20}(1 + 4 + 9 + 16 + 16 \times 2) = \frac{31}{10}$$

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Question 6: [7%, Work-out question]
Define the following two signals:

$$x(t) = e^{-3(t-1)}U(t-1) \tag{5}$$
$$h(t) = e^{-2t}U(t) \tag{6}$$

Find the expression of

$$y(t) = x(t) * h(t). \tag{7}$$

Hint: Table 4.2 may be useful when answering this question.

Answer:

Using Table 4.2 the transform pair for

$$e^{-at}u(t) \quad Re\{a\} > 0 \longleftrightarrow \frac{1}{a+j\omega}$$

$$x(t) \overset{FT}{\longleftrightarrow} X(j\omega) = \frac{1}{3+j\omega} e^{-j\omega}$$

$$h(t) \overset{FT}{\longleftrightarrow} H(j\omega) = \frac{1}{2+j\omega}$$

By the convolution property:

$$Y(j\omega) = X(j\omega) \cdot H(j\omega)$$
$$= \frac{1}{3+j\omega} \cdot \frac{1}{2+j\omega} e^{-j\omega}$$

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This sheet is for Question 6.

$$= \left(\frac{-1}{3+j\omega} + \frac{1}{2+j\omega} \right) e^{-j\omega}$$

Using the transform pair again

$$y(t) = \left(e^{-2(t-1)} - e^{-3(t-1)} \right) u(t-1)$$

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This sheet is for Question 6.

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Question 7: [8%, Work-out question]

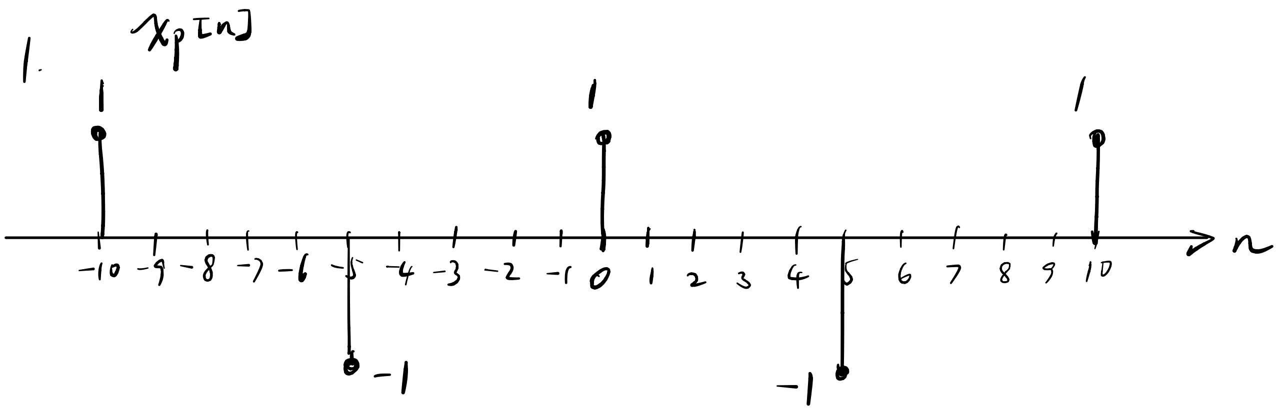
Consider a DT signal $x[n] = \cos(0.2\pi n)$. Define $p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 5k]$.

1. [3%] Define $x_p[n] = x[n] \cdot p[n]$. Plot $x_p[n]$ for the range of $-10 \leq n \leq 10$.
2. [5%] Find the DTFS coefficients a_k of the DT signal $x_p[n]$.

   Hint: If you do not know the answer of Q7.1, you can assume $x_p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 2 - 5k]$. You will receive 3.5 points if your answer is correct.

Answer:

1.



2. $$x_p[n] = \sum_{k=-\infty}^{+\infty} \delta[n - 10k] - \sum_{k=-\infty}^{+\infty} \delta[n-5-10k]$$

Using the impulse train transform pair.

$$X_p(j\omega) = \frac{2\pi}{10} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{10}\right) - \frac{2\pi}{10} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{10}\right) e^{-j5\omega}$$

$$= \frac{2\pi}{10} \sum_{k=-\infty}^{+\infty} \left(1 - e^{-j5\frac{2\pi}{10}k}\right) \delta\left(\omega - \frac{2\pi k}{10}\right)$$

$$\therefore a_k = \frac{1}{10}\left(1 - e^{-j\pi k}\right)$$

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Question 8: [15%, Multiple-choice question] Consider two signals

$$h_1(t) = e^{\int_{-t}^{t} \cos(s)+s^2 \sin(s) ds} \tag{8}$$

and

$$h_2[n] = \sum_{k=0}^{\infty} 2^{-n} e^{j1000n} U[n-5k] \tag{9}$$

Yes 1. [1.25%] Is $h_1(t)$ periodic?

No 2. [1.25%] Is $h_2[n]$ periodic?

neither 3. [1.25%] Is $h_1(t)$ even or odd or neither?

neither 4. [1.25%] Is $h_2[n]$ even or odd or neither?

Yes 5. [1.25%] Is $h_1(t)$ of finite power?

Yes 6. [1.25%] Is $h_2[n]$ of finite power?

Suppose the above two signals are also the impulse responses of two LTI systems: System 1 and System 2, respectively.

No 1. [1.25%] Is System 1 memoryless?

No 2. [1.25%] Is System 2 memoryless?

No 3. [1.25%] Is System 1 causal?

Yes 4. [1.25%] Is System 2 causal?

No 5. [1.25%] Is System 1 stable?

Yes 6. [1.25%] Is System 2 stable?

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This sheet is for Question 8.

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2. \quad (3.6$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \|a_k\| = \|a_{-k}\| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}e\{X(j\omega)\}$ $j\mathcal{I}m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & T_1 < \lvert t \rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & \lvert t \rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix}X(e^{j\omega})\\Y(e^{j\omega})\end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$		

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

5.7.1 Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients a_k of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence a_k in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence a_k are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \quad k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \quad k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—