Midterm #3 of ECE 301-003 and OL1, (CRN: 11474 and 26295)
8–9pm, Wednesday, April 7, 2021, Online Exam.

1. Enter your student ID number, and signature in the space provided on this page **now!**
2. This is a closed book exam.
3. This exam contains exclusively work-out questions. You have one hour to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [18%, Work-out question] Consider two discrete-time periodic signals

$$x[n] = \sin(\frac{6\pi n}{5}) \quad (1)$$
$$y[n] = \cos(\frac{2\pi n}{5}) \quad (2)$$
$$\quad (3)$$

Denote the DTFS coefficients of $x[n]$ by a_k and denote the DTFS coefficients of $y[n]$ by b_k. Answer the following questions.

1. [2%] Find the period of $x[n]$.
2. [6%] Plot the values of a_k for the range of $k = -5$ to 5.
3. [10%] Define $z[n] = x[n] \cdot y[n]$ and denote the corresponding DTFS coefficients as c_k. Plot the values of c_k for the range of $k = -5$ to 5.

   Hint: If you do not know the answer to this question, please write down the expression of c_3 in terms of a_k and b_k formulas. Namely, you can just assume a_k and b_k are known numbers and use them to find your c_3 value. You will receive 7 points if your answer is correct.

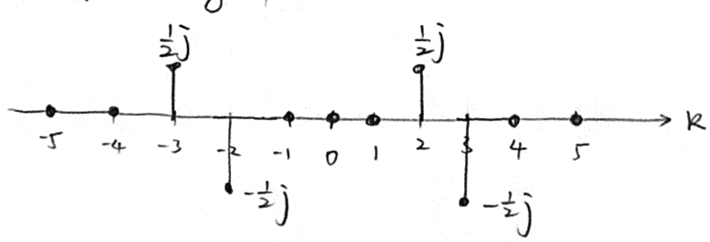
Answer:

1. $\frac{6\pi}{5} N = 2\pi m$, $m \in \mathbb{Z}$ $\Rightarrow$ $N = \frac{5}{3} m$ the fundamental period is $N = 5$.

2. $x[n] = \frac{1}{2j} e^{j \cdot 3 \frac{2\pi}{5} n} - \frac{1}{2j} e^{-j \cdot 3 \frac{2\pi}{5} n}$

$a_{-3} = \frac{1}{2}j$ and $a_3 = -\frac{1}{2}j$

With periodicity of 5, we have



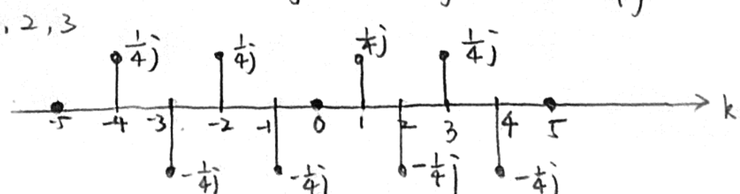
3. By multiplication property

$a_2 = \frac{1}{2}j$ $\quad a_3 = -\frac{1}{2}j$ $\quad a_k = 0$ for $k = 0, 1, 4$

$b_1 = \frac{1}{2}$ $\quad b_4 = \frac{1}{2}$ $\quad b_k = 0$ for $k = 0, 2, 3$

$C_0 = \sum_{h=0}^{4} a_h b_{0-h} = 0$ $\quad C_1 = \sum_{h=0}^{4} a_h b_{1-h} = \frac{1}{4}j$

$\Rightarrow C_2 = -\frac{1}{4}j$ $\quad C_3 = \frac{1}{4}j$ $\quad C_4 = -\frac{1}{4}j$



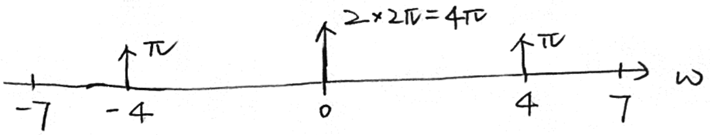
Question 2: [22%, Work-out question] Consider a CT-LTI system with the impulse response as follows.

$$h(t) = \begin{cases} \pi & \text{if } -\pi \leq t \leq \pi \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

We use $y(t)$ to denote the output of the system when the input is $x(t) = 2 + \cos(4t)$. Answer the following questions:

1. [6%] Let $X(j\omega)$ denote the CTFT of the input signal $x(t)$. Plot $X(j\omega)$ for the range of $-3 \leq \omega \leq 3$.
2. [2%] What is the definition of the *frequency response* of an LTI system?
3. [10%] Let $H(j\omega)$ denote the CTFT of $h(t)$. Plot $H(j\omega)$ for the range of $-3 \leq \omega \leq 3$. Please carefully mark all the special points of your figure.
4. [4%] Find the expression of $y(t)$.

Answer : 1. x(t)=0.5e^{j4t}+0.5e^{-j4t}+2; Thus X(jw)=\pi * \delta(w-4)+\pi * \delta(w+4) + 4\pi \delta(w).

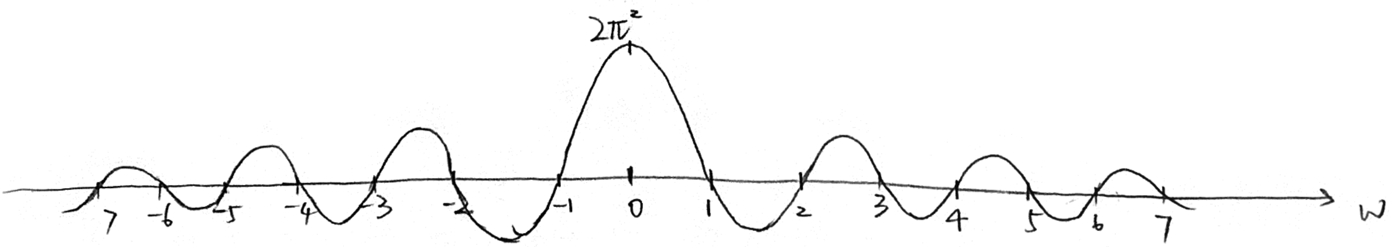


2. The freqney response of LTI system h(t) is

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$$

3. From transform pair in Table 4.2

$$H(j\omega) = \pi \cdot \frac{2 \sin \pi w}{w}$$



4. By convolution property

$$x(t) * h(t) \xleftrightarrow{F} X(j\omega) H(j\omega)$$

$$Y(j\omega) = 4\pi \cdot 2\pi^2 \delta(\omega) = 8\pi^3 \delta(\omega)$$

$$y(t) = 4\pi^2.$$

Question 3: [16%, Work-out question] We know that if

$$x(t) = e^{-|t|} \tag{5}$$

the corresponding CTFT is

$$X(j\omega) = \frac{2}{1+\omega^2} \tag{6}$$

Answer the following questions:

1. [8%] Find the value of $\int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2}\right)^2 d\omega$.
2. [8%] Find the value of $\int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2}\right) \cdot e^{j3\omega} d\omega$.

Answer:

1. By Parseval's relation:

$$\int_{-\infty}^{+\infty} \left(\frac{2}{1+w^2}\right)^2 dw = 2\pi \int_{-\infty}^{+\infty} e^{-|t| \cdot 2} dt$$

$$= 4\pi \int_0^{+\infty} e^{-2t} dt$$

$$= 4\pi \left. \frac{e^{-2t}}{-2} \right|_{t=0}^{+\infty}$$

$$= +2\pi$$

2. By the synthesis equation

$$x(3) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(jw) e^{j3w} dw$$

$$\Rightarrow \int_{-\infty}^{+\infty} X(jw) e^{j3w} dw = 2\pi x(3) = 2\pi e^{-3}$$

Question 4: [14%, Work-out question] Consider the following periodic CT signal $x(t)$

$$x(t) = \begin{cases} 1 & -1.5 \le t \le 1.5 \\ 0 & 1.5 < t < 6.5 \\ \text{periodic with period 8} & \end{cases} \quad (7)$$

Plot $X(jw)$ for the range of $-\frac{\pi}{3} \le \omega \le \frac{\pi}{3}$.

Hint 1: This type of computation is termed the *generalized CTFT* in the lecture.

Hint 2: If you do not know how to find the CTFT of $x(t)$, you should find the CTFS of $x(t)$ instead. You will receive 9 points if your answer is correct.

Hint 3: The CTFS of $x(t)$ actually helps you find the CTFT of $x(t)$.

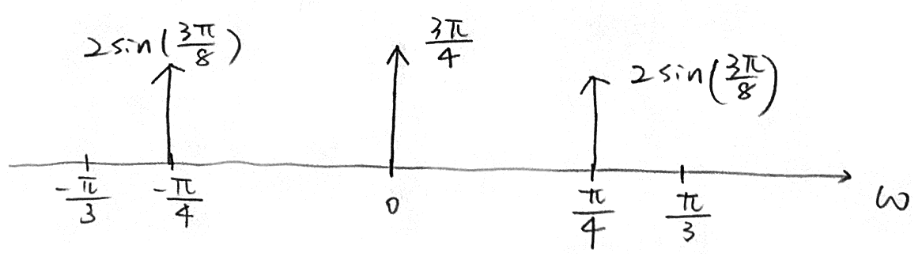
Answer: By table 4.2 periodic square wave,

we have $T_1 = 1.5$ and $T = 8$

$$X(jw) = \sum_{k=-\infty}^{+\infty} \frac{2\sin(k\frac{2\pi}{8} \cdot 1.5)}{k} \delta(w - \frac{2\pi}{8}k)$$

$$= \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{2\sin(\frac{3\pi}{8}k)}{k} \delta(w - \frac{\pi}{4}k)$$

$$X(j0) = 2\pi \cdot \frac{2T_1}{T} = \frac{3\pi}{4} \quad \text{for } k=0$$



Question 5: [12%, Work-out question] Consider a discrete time signal $x[n]$ and we know that the DTFT $X(e^{j\omega})$ satisfies

$$X(e^{j\omega}) = \cos(\omega) + j\sin(3\omega) \tag{8}$$

Plot the signal $x[n]$ for the range of $-5 \le n \le 5$.

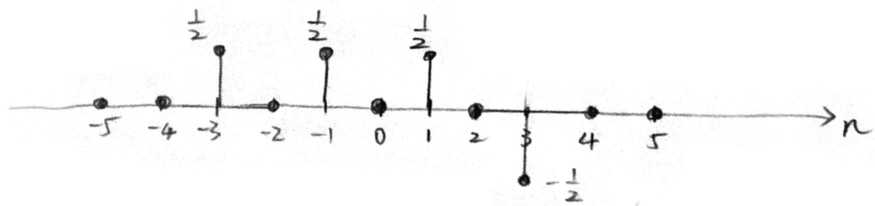
Hint: The DTFT *analysis formula* $X(e^{j\omega}) = \sum_{n=-\infty} x[n]e^{-j\omega n}$ may be useful for solving this question.

Answer: $X(e^{jw}) = \frac{1}{2}e^{jw} + \frac{1}{2}e^{-jw} + \frac{j}{2j}e^{j3w} - \frac{j}{2j}e^{-j3w}$

From the DTFT analysis formula, we have

$X[-1] = \frac{1}{2}$ $\quad X[1] = \frac{1}{2}$ $\quad X[-3] = \frac{1}{2}$ $\quad x[3] = -\frac{1}{2}$

$x[n] = 0$ otherwise



Question 6: [18%, Work-out question] Consider the following sequential operations. The input signal is $x(t) = \frac{\sin(20\pi t)}{\pi t}$. Step 1: We multiply $x(t)$ by $\cdot \cos(10\pi t)$. That is,

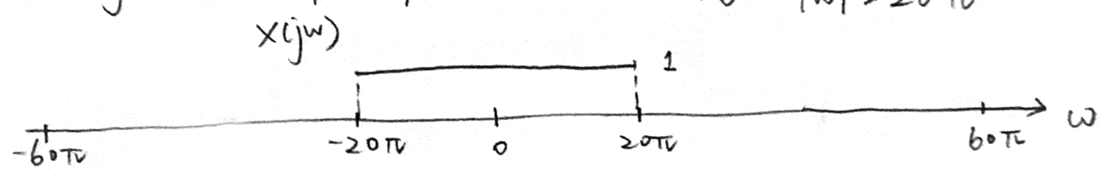
$$y(t) = x(t) \cdot \cos(10\pi t).$$

Step 2: We pass $y(t)$ through ~~through~~ a low pass filter with cutoff frequency 10Hz. Denote the output by $z(t)$.

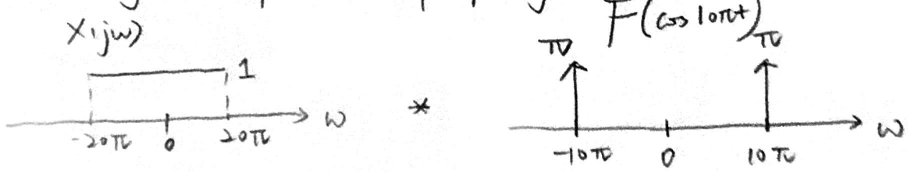
1. [5%] Plot the CTFT $X(j\omega)$ for the range of $\omega = -60\pi$ to 60π.
2. [5%] Plot the CTFT $Y(j\omega)$ for the range of $\omega = -60\pi$ to 60π.
3. [4%] Plot the CTFT $Z(j\omega)$ for the range of $\omega = -60\pi$ to 60π.
4. [4%] Find out the expression of $z(t)$.

Answer:

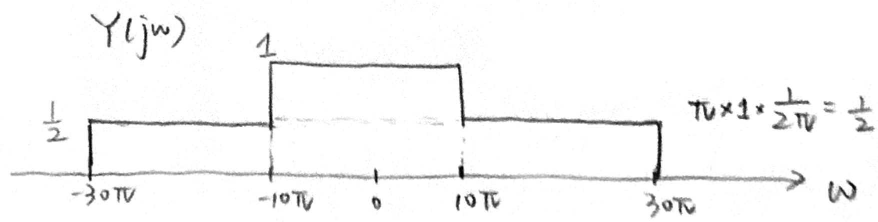
1. By CTFT transform pair $X(j\omega) = \begin{cases} 1 & |\omega| < 20\pi \\ 0 & |\omega| > 20\pi \end{cases}$



2. By multiplication property

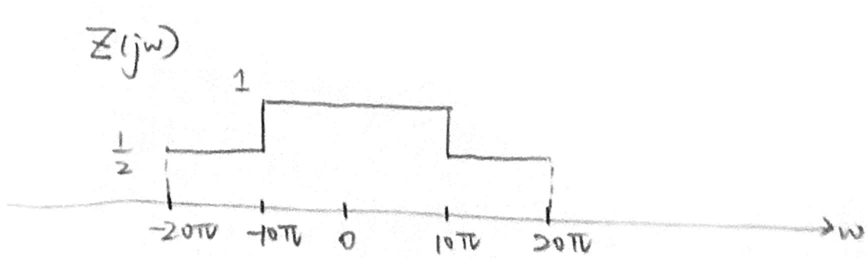


$\Rightarrow$



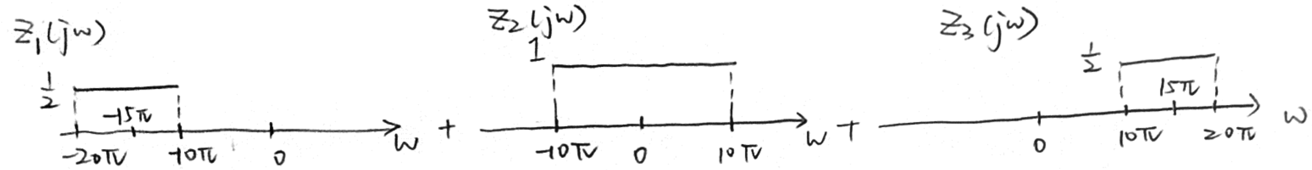
$\pi \times 1 \times \frac{1}{2\pi} = \frac{1}{2}$

3.



4. From 3, we have

$Z(jw) =$



By CTFT transform pair, the frequency shifting property, and the linearity of CTFT, we have

$$z(t) = z_1(t) + z_2(t) + z_3(t)$$

$$= \frac{1}{2}e^{-j15\pi t}\frac{\sin(5\pi t)}{\pi t} + \frac{\sin(10\pi t)}{\pi t} + \frac{1}{2}e^{j15\pi t}\frac{\sin(5\pi t)}{\pi t}$$

$$= \frac{\sin(10\pi t)}{\pi t} + \frac{\sin(5\pi t)}{\pi t}\cdot\cos(15\pi t)$$

$$= \frac{\sin(10\pi t)}{2\pi t} + \frac{\sin(20\pi t)}{2\pi t}$$