Midterm #2 of ECE 301-003 and OL1, (CRN: 11474 and 26295)
8–9pm, Wednesday, March 10, 2021, Online Exam.

1. Enter your student ID number, and signature in the space provided on this page **now!**
2. This is a closed book exam.
3. This exam contains multiple-choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [22%, Work-out question]

1. [2%] What is the definition of the *step response*?

Consider a DT-LTI system. We know that for this system, if the input is

$$x[n] = \begin{cases} 1 & \text{if } -3 \leq n \\ 0 & \text{if } n \leq -4 \end{cases} \tag{1}$$

then the output is

$$y[n] = \begin{cases} 2 - 2^{-(n+3)} & \text{if } -3 \leq n \\ 0 & \text{if } n \leq -4 \end{cases} \tag{2}$$

2. [6%] Denote the *step response* of this system by $h_{\text{step}}[n]$. Plot $h_{\text{step}}[n]$ for the range of $-4 \leq n \leq 4$.

   Hint 1: For this sub-question, there is no need to write down the mathematical expression of $h_{\text{step}}[n]$. We only ask you to plot $h_{\text{step}}[n]$.

3. [8%] Denote the *impulse response* of this system by $h[n]$. Plot $h[n]$ for the range of $-4 \leq n \leq 4$.

   Hint 2: For this sub-question, there is no need to write down the mathematical expression of $h[n]$. We only ask you to plot $h[n]$.

   Hint 3: If you do not know the answer to the previous sub-question, you may assume the step response is $h_{\text{step}}[n] = \cos(\frac{\pi n}{2})(U[n-4] - U[n])$ and use it to solve $h[n]$. You will receive full credit if your answer is correct.

4. [6%] Is this LTI system stable? Please carefully explain your answer/reason. A yes/no answer without any justification will receive zero credit.

   Hint 4: If you do not know the $h[n]$ in the previous sub-question, you can answer how one can use the impulse response to check whether a system is stable. If your method on "how to check stability" is correct, you would still receive 4 points even if you did not carry out your own method. Unfortunately, you may not use the $h_{\text{step}}[n]$ in Hint 3 (i.e., Q1.3) when solving Q1.4.

Answer

1. A step response is the output of a LTI system given a unit step signal as the input.

$$h_{step}(t) = \underbrace{h(t)}_{\text{impulse response}} * \underbrace{u(t)}_{\text{input}}$$

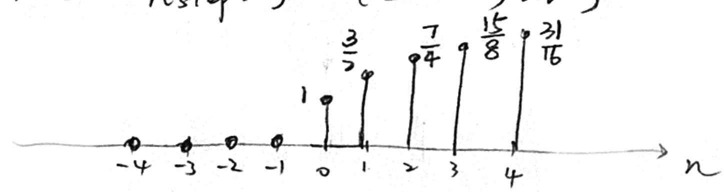
2. As given in the question

$$x[n] = u[n+3] \longrightarrow \boxed{LTI} \longrightarrow y[n] = (2-2^{-(n+3)})u[n+3]$$

By time-invariance property

$$x[n] = u[n] \rightarrow \boxed{LTI} \rightarrow y[n] = (2-2^{-n})u[n]$$

Thus $\quad h_{step}[n] = (2-2^{-n})u[n]$



3. From (2), we know

$$\delta[n] = u[n] - u[n-1]$$

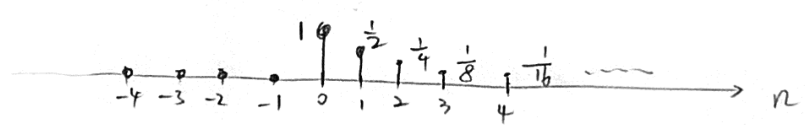
By linearity: $h[n] = h_{step}[n] - h_{step}[n-1]$

$$= (2-2^{-n})u[n] - (2-2^{-(n-1)})u[n-1]$$

$$= 2(u[n]-u[n-1]) - 2^{-n}(u[n]-2u[n-1])$$

$$= 2\delta[n] - 2^{-n}\delta[n] + 2^{-n}u[n-1]$$

$$= 2^{-n}u[n].$$



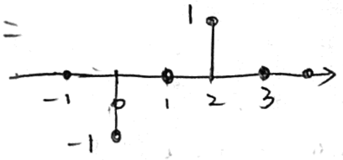
4. It is stable, because $\sum_{k=-\infty}^{+\infty} |h_k| = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = 2 < \infty.$

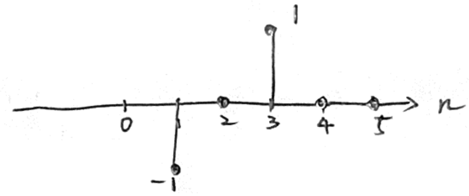
Alternative

3.

$$\delta[n] = u[n] - u[n-1]$$

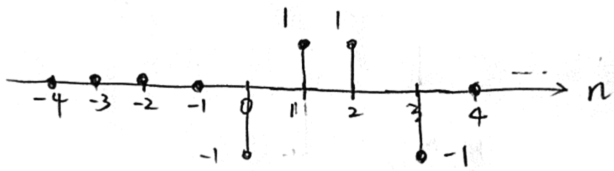
By linearity $\quad h[n] = h_{step}[n] - h_{step}[n-1]$

$=$ 

$-$ 

$$= -\delta[n] + \delta[n-1] + \delta[n-2] - \delta[n-3]$$

$h[n]$



4. Given a LTI system with impulse response $h[n]$,

if $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$, the system is stable

Generally, to have a stable system,

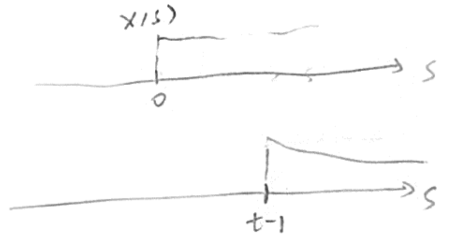
the output is bound for any bounded input.

Question 2: [12%, Work-out question] Consider three signals:

$$x(t) = e^{j3t} \cdot U(t) \tag{3}$$
$$z(t) = 2^{\pi t} \cdot U(-t+1) \tag{4}$$

where $U(t)$ is the unit step signal. Find the expression $h(t) = x(t) * z(t)$.

Hint: Your answer can be of the form like $\frac{e^{3t} - e^{\pi t}}{1 - e^{-1.5t}}$. There is no need to further simplify it.



Answer.

$$x(s) = e^{j3s} \cdot u(s)$$

$$z(t-s) = 2^{\pi(t-s)}\, U(-t+s+1)$$

$$h(t) = \int_{s=-\infty}^{+\infty} x(s) z(t-s)\, ds$$

Case 1 when $t-1 < 0 \Rightarrow t < 1$

$$h(t) = \int_{s=0}^{+\infty} e^{j3s}\, 2^{\pi(t-s)}\, ds$$

$$= 2^{\pi t} \int_{s=0}^{+\infty} (e^{3j} 2^{-\pi})^s\, ds$$

$$= 2^{\pi t} \left. \frac{(e^{3j}2^{-\pi})^s}{\ln(e^{3j}2^{-\pi})} \right|_{s=0}^{+\infty}$$

$$= \frac{-2^{\pi t}}{3j - \pi \ln 2}$$

Case 2 when $t-1 \geq 0 \Rightarrow t \geq 1$

$$h(t) = 2^{\pi t} \int_{s=t-1}^{+\infty} (e^{3j} 2^{-\pi})^s\, ds$$

$$= 2^{\pi t} \left. \frac{(e^{3j}2^{-\pi})^s}{3j - \pi \ln 2} \right|_{s=t-1}^{\infty}$$

$$= \frac{-2^{\pi t} \cdot (e^{3j}2^{-\pi})^{t-1}}{3j - \pi \ln 2}$$

$$= \frac{-e^{3j(t-1)} \cdot 2^{+\pi}}{3j - \pi \ln 2}$$

Thus

$$h(t) = \begin{cases} \dfrac{-2^{\pi t}}{3j - \pi \ln 2} & t < 1 \\[2ex] \dfrac{(-2^{\pi} e^{-3j})\, e^{3jt}}{3j - \pi \ln 2} & t \geq 1 \end{cases}$$

Question 3: [12%, Work-out question]

Consider a DT-LTI system with the impulse response being $h[n] = 3 \cdot e^{-|n|}$. Let $y[n]$ denote the output when the input is $x[n] = e^{j0.5\pi n}$. Find the absolute value of $y[3]$, i.e., find $|y[3]|$, and also find the angle of $y[3]$, i.e., find $\angle y[3]$.

Hint: If you may need the following formula:

$$\text{if } |r| < 1, \text{ then } \sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \tag{5}$$

$$\text{if } r \neq 1, \text{ then } \sum_{k=1}^{K} a \cdot r^{k-1} = \frac{a \cdot (1-r^K)}{1-r} \tag{6}$$

Answer:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

$$y[3] = \sum_{k=-\infty}^{+\infty} e^{j\frac{\pi}{2}k}\, 3e^{-|3-k|}$$

$$= 3\sum_{k=-\infty}^{2} e^{j\frac{\pi}{2}k} e^{-(3-k)} + 3\sum_{k=3}^{\infty} e^{j\frac{\pi}{2}k} e^{3-k}$$

$$= 3e^{-3}\sum_{k=-\infty}^{2} e^{(j\frac{\pi}{2}+1)k} + 3e^{3}\sum_{k=3}^{\infty} e^{j\frac{\pi}{2}k} e^{-k}$$

$$= 3e^{-3}\sum_{k=-2}^{+\infty} e^{-(j\frac{\pi}{2}+1)k} + 3e^{3}\sum_{k=3}^{\infty} e^{(j\frac{\pi}{2}-1)k}$$

$$= 3e^{-3}\frac{e^{2(j\frac{\pi}{2}+1)}}{1-e^{-(j\frac{\pi}{2}+1)}} + 3e^{3}\frac{e^{3(j\frac{\pi}{2}-1)}}{1-e^{(j\frac{\pi}{2}-1)}}$$

$$= \frac{-3e^{-1}}{1+je^{-1}} + \frac{-3j}{1-je^{-1}}$$

$$= \frac{-3j+3je^{-2}}{1+e^{-2}}$$

$$|y[3]| = \frac{3(1-e^{-2})}{1+e^{-2}} \qquad \angle y[3] = \frac{3\pi}{2}$$

Question 4: [22%, Work-out question] Consider two periodic CT signals

$$x(t) = e^{j\sqrt{2}\cdot\pi t} \tag{7}$$
$$y(t) = \sin(\sqrt{2}\cdot\pi t) \tag{8}$$

1. [6%] Find the period T_Y and the Fourier series coefficients b_k of $y(t)$.

   Hint 1: The following formulas may be useful:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{9}$$
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{10}$$

2. [8%] Define $z(t) = x(t - \frac{1}{\sqrt{2}})$. Find the period T_Z and the Fourier series coefficients c_k of $z(t)$. Please write down explicitly the values of c_{-1}, c_0, and c_1 by plugging in the k values to be -1, 0, and 1, respectively.

3. [8%] Define $w(t) = z(t)\cdot y(t)$. Find the period T_W and the Fourier series coefficients d_k of $w(t)$.

   Hint 2: If you do not know the answers to the previous two sub-questions, you can assume $b_k = \sum_{m=-1}^{1}\delta[k-2m]$ and $c_k = U[n+1] - U[n-2]$ and $T_Y = T_Z = 8$ when solving Q4.3. You will receive full credit if your answer is correct.

Answer 1. $y(t) = \dfrac{e^{j\sqrt{2}\pi t} - e^{-j\sqrt{2}\pi t}}{2j}$ with $T_Y = \dfrac{2\pi}{\sqrt{2}\pi} = \sqrt{2}$

Thus $b_{-1} = +\frac{1}{2}j$ $b_1 = -\frac{1}{2}j$, $b_k = 0$ for $k \neq \pm 1$

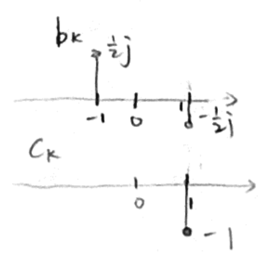
2 $T_Z = T_X = \dfrac{2\pi}{\sqrt{2}\pi} = \sqrt{2}$

CTFS of $x(t)$: $a_1 = 1$, $a_k = 0$ for $k \neq 1$

By time-shifting property: $C_k = a_k \cdot e^{-jk\omega_0/\sqrt{2}} = a_k e^{-jk\pi}$

Thus $C_{-1} = C_0 = 0$, $C_1 = a_1 \cdot e^{-j\pi} = -1$, $C_k = 0$ otherwise

3. $T_W = LCM(T_Y, T_Z) = \sqrt{2}$.



By multiplication property: $d_k = b_k * C_k$

$d_0 = b_{-1}\cdot C_1 = -\frac{1}{2}j$ $d_2 = b_1 C_1 = \frac{1}{2}j$

$d_k = 0$ for $k \neq 0, 2$

3. Alternative

$$b_k = \delta[k+2] + \delta[k] + \delta[k-2]$$

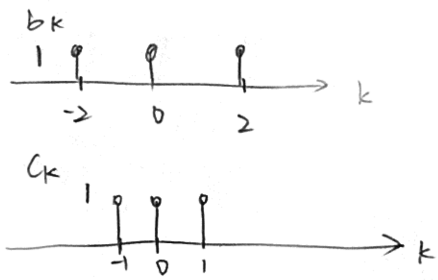
$$c_k = \delta[k+1] + \delta[k] + \delta[k-1]$$

$$T_y = T_z = 8$$

$$w(t) = z(t) \cdot y(t)$$

$$T_w = LCM(T_y, T_z) = 8$$

$$d_k = b_k * c_k$$



$$d_k = [1, 1, 2, 1, 2, 1, 1]$$

for $k = -3, -2, -1, 0, 1, 2, 3,$

and $d_k = 0$ otherwise

Question 5: [10%, Work-out question] Consider the following DT periodic signal

$$x[n] = \begin{cases} e^{j\pi n} & \text{if } 1 \leq n \leq 10 \\ 0 & \text{if } 11 \leq n \leq 20 \\ \text{periodic with period } 20 & \end{cases} \tag{11}$$

Define a_k as the FS coefficients of $x[n]$.

Find the value of a_{10}.

Hint: Your answer can be of the form like $\frac{e^{3n}-e^{\pi n}}{1-e^{-1.5n}}$. There is no need to further simplify it.

Answer:

$$a_k = \frac{1}{N}\sum_{<N>} x[n]e^{-jk\omega_0 n} \quad \text{where } \omega_0 = \frac{2\pi}{20} = \frac{\pi}{10}$$

$$a_{10} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n}$$

$$= \frac{1}{N}\sum_{n=1}^{10} e^{j\pi n} e^{-j\pi n}$$

$$= \frac{10}{20}$$

$$= \frac{1}{2}$$

Question 6: [20%, Multiple Choices]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \int_{-\infty}^{t} e^{-|x(s)|\cdot s} ds \tag{12}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = \sum_{k=-\infty}^{\infty} e^{|k|} \sin(\pi \cdot 2^{|k|-1}) x[n-k] \tag{13}$$

Answer the following questions

1. [4%] Is System 1 memoryless? Is System 2 memoryless?
2. [4%] Is System 1 causal? Is System 2 causal?
3. [4%] Is System 1 stable? Is System 2 stable?
4. [4%] Is System 1 linear? Is System 2 linear?
5. [4%] Is System 1 time-invariant? Is System 2 time-invariant?

Answer: System 1.

with memory,

causal,

unstable (eg: input $x(t) = 1$)

non-linear

time-varying (① input $x(t-t_0)$: $\tilde{y}(t) = \int_{s=-\infty}^{t} e^{-|x(s-t_0)|s} ds$

let $s' = s - t_0$

$= \int_{s'=-\infty}^{t-t_0} e^{-|x(s')|(s'+t_0)} ds$

② delay $y_1(t)$ by t_0:

$y_1(t-t_0) = \int_{s=-\infty}^{t-t_0} e^{-|x(s)|s} ds$

① ≠ ②)

System 2

$\sin(\pi \cdot 2^{|k|-1})$

$= \begin{cases} 1 & k=0 \\ 0 & \text{otherwise} \end{cases}$

$y_2[n] = x[n]$

memoryless

causal

stable

linear

time-invariant.