Midterm #2 of ECE 301-003 and OL1, (CRN: 11474 and 26295)
8–9pm, Wednesday, March 10, 2021, Online Exam.

1. Enter your student ID number, and signature in the space provided on this page **now!**
2. This is a closed book exam.
3. This exam contains multiple-choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [22%, Work-out question]

1. [2%] What is the definition of the *step response*?

Consider a DT-LTI system. We know that for this system, if the input is

$$x[n] = \begin{cases} 1 & \text{if } -3 \leq n \\ 0 & \text{if } n \leq -4 \end{cases} \tag{1}$$

then the output is

$$y[n] = \begin{cases} 2 - 2^{-(n+3)} & \text{if } -3 \leq n \\ 0 & \text{if } n \leq -4 \end{cases} \tag{2}$$

2. [6%] Denote the *step response* of this system by $h_{\text{step}}[n]$. Plot $h_{\text{step}}[n]$ for the range of $-4 \leq n \leq 4$.

   Hint 1: For this sub-question, there is no need to write down the mathematical expression of $h_{\text{step}}[n]$. We only ask you to plot $h_{\text{step}}[n]$.

3. [8%] Denote the *impulse response* of this system by $h[n]$. Plot $h[n]$ for the range of $-4 \leq n \leq 4$.

   Hint 2: For this sub-question, there is no need to write down the mathematical expression of $h[n]$. We only ask you to plot $h[n]$.

   Hint 3: If you do not know the answer to the previous sub-question, you may assume the step response is $h_{\text{step}}[n] = \cos(\frac{\pi n}{2})(U[n-4] - U[n])$ and use it to solve $h[n]$. You will receive full credit if your answer is correct.

4. [6%] Is this LTI system stable? Please carefully explain your answer/reason. A yes/no answer without any justification will receive zero credit.

   Hint 4: If you do not know the $h[n]$ in the previous sub-question, you can answer how one can use the impulse response to check whether a system is stable. If your method on "how to check stability" is correct, you would still receive 4 points even if you did not carry out your own method. Unfortunately, you may not use the $h_{\text{step}}[n]$ in Hint 3 (i.e., Q1.3) when solving Q1.4.

Question 2: [12%, Work-out question] Consider two signals:

$$x(t) = e^{j3t} \cdot U(t) \tag{3}$$

$$z(t) = 2^{\pi t} \cdot U(-t+1) \tag{4}$$

where $U(t)$ is the unit step signal. Find the expression $h(t) = x(t) * z(t)$.

Hint: Your answer can be of the form like $\frac{e^{3t}-e^{\pi t}}{1-e^{-1.5t}}$. There is no need to further simplify it.

Question 3: [14%, Work-out question]

Consider a DT-LTI system with the impulse response being $h[n] = 3 \cdot e^{-|n|}$. Let $y[n]$ denote the output when the input is $x[n] = e^{j0.5\pi n}$. Find the absolute value of $y[3]$, i.e., find $|y[3]|$, and also find the angle of $y[3]$, i.e., find $\angle y[3]$.

Hint: If you may need the following formula:

$$\text{if } |r| < 1, \text{ then } \sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \tag{5}$$

$$\text{if } r \neq 1, \text{ then } \sum_{k=1}^{K} a \cdot r^{k-1} = \frac{a \cdot (1 - r^K)}{1-r} \tag{6}$$

Question 4: [22%, Work-out question] Consider two periodic CT signals

$$x(t) = e^{j\sqrt{2}\cdot\pi t} \tag{7}$$

$$y(t) = \sin(\sqrt{2}\cdot\pi t) \tag{8}$$

1. [6%] Find the period T_Y and the Fourier series coefficients b_k of $y(t)$.

   Hint 1: The following formulas may be useful:

   $$\cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2} \tag{9}$$

   $$\sin(\theta) = \frac{e^{j\theta}-e^{-j\theta}}{2j} \tag{10}$$

2. [8%] Define $z(t) = x(t - \frac{1}{\sqrt{2}})$. Find the period T_Z and the Fourier series coefficients c_k of $z(t)$. Please write down explicitly the values of c_{-1}, c_0, and c_1 by plugging in the k values to be -1, 0, and 1, respectively.

3. [8%] Define $w(t) = z(t)\cdot y(t)$. Find the period T_W and the Fourier series coefficients d_k of $w(t)$.

   Hint 2: If you do not know the answers to Q4.1 and Q4.2, you can assume $b_k = \sum_{m=-1}^{1}\delta[k-2m]$ and $c_k = U[k+1]-U[k-2]$ and $T_Y = T_Z = 8$ when solving Q4.3. You will receive full credit if your answer is correct.

Question 5: [10%, Work-out question] Consider the following DT periodic signal

$$x[n] = \begin{cases} e^{j\pi n} & \text{if } 1 \leq n \leq 10 \\ 0 & \text{if } 11 \leq n \leq 20 \\ \text{periodic with period 20} & \end{cases} \tag{11}$$

Define a_k as the FS coefficients of $x[n]$.

Find the value of a_{10}.

Hint: Your answer can be of the form like $\frac{e^{3n}-e^{\pi n}}{1-e^{-1.5n}}$. There is no need to further simplify it.

Question 6: [20%, Multiple Choices]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \int_{-\infty}^{t} e^{-|x(s)|\cdot s} ds \tag{12}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = \sum_{k=-\infty}^{\infty} e^{|k|} \sin(\pi \cdot 2^{|k|-1}) x[n-k] \tag{13}$$

Answer the following questions

1. [4%] Is System 1 memoryless? Is System 2 memoryless?
2. [4%] Is System 1 causal? Is System 2 causal?
3. [4%] Is System 1 stable? Is System 2 stable?
4. [4%] Is System 1 linear? Is System 2 linear?
5. [4%] Is System 1 time-invariant? Is System 2 time-invariant?

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	Ta_kb_k
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2. \quad (3.6$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$
	Parseval's Relation for Periodic Signals $\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x[n] \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$	