Question 1: [15%, Work-out question] Consider a CT signal:

$$x(t) = e^{-3|t|} \tag{1}$$

We construct another signal

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\omega)e^{j\omega t}d\omega \tag{2}$$

1. [11%] What is the expression of $y(t)$?
   Hint 1: The absolute value of $3j$ is $|3j| = 3$.

2. [4%] What is the even part of $y(t)$? What is the odd part of $y(t)$? Please write down the expressions of both parts separately.
   Hint 2: You may want to write $\frac{1}{b+c\cdot j} = \frac{b-c}{b^2+c^2}$. This would make the expression simpler.

Hint 3: If you do not know the answers of Q1.1 and Q1.2, you can write down the odd part of a DT signal $y[n] = U[n-1]$ and plot it for the range of $-5 \leq n \leq 5$. If your answer is correct, you will receive 6 points.

Note that if you know the answer of Q1.1, you cannot use this alternative question. The alternative question is only for students who do not know how to solve Q1.1.

Answer: 1.

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-3|j\omega|}e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{\omega=0}^{+\infty} e^{-3\omega}e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{\omega=-\infty}^{0} e^{3\omega}e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\frac{e^{(jt-3)\omega}}{jt-3}\Big|_{\omega=0}^{\infty} + \frac{1}{2\pi}\frac{e^{(jt+3)\omega}}{jt+3}\Big|_{\omega=-\infty}^{0}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jt-3}(0-1) + \frac{1}{2\pi}\frac{1}{jt+3}(1-0)$$

$$= \frac{1}{2\pi}\frac{-6}{-t^2-9}$$

$$= \frac{3}{\pi t^2+9\pi}$$

2. $$y_{even}(t) = \frac{y(t) + y(-t)}{2}$$

$$= \frac{3}{\pi t^2 + 9\pi}$$

$$y_{odd}(t) = \frac{y(t) - y(-t)}{2}$$

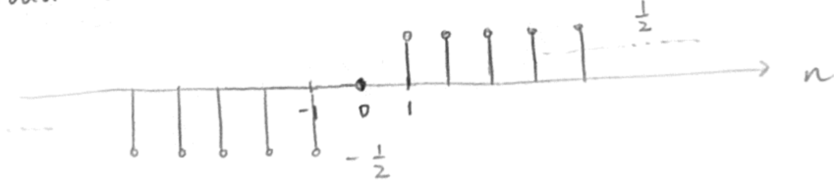
$$= 0.$$

Alternative:

$$y_{odd}[n] = \frac{y[n] - y[-n]}{2}$$

$$= \frac{u[n-1] - u[-n-1]}{2}$$

$y_{odd}[n]$:



Question 2: [20%, Work-out question]

Define two DT signals:

$$x[n] = \begin{cases} 9^n & \text{if } n \le 3 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and

$$h[n] = \begin{cases} 3^n & \text{if } n \le -2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Compute the expression of the following integral

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[2n-2k] \tag{5}$$

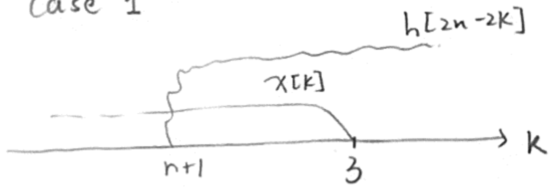
Hint: If you do not know the answer to this question, write down the value of the summation $\sum_{k=19}^{221} \sqrt{2}$. You will receive 4 points if your answer is correct.

Answer.

$$x[k] = \begin{cases} 9^k & \text{if } k \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$h[2n-2k] = \begin{cases} 3^{2n-2k} & \text{if } 2n-2k \le -2 \Rightarrow k \ge n+1 \\ 0 & \text{otherwise} \end{cases}$$

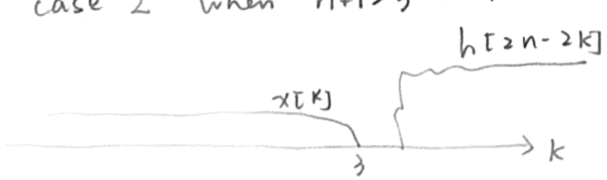
case 1 when $n+1 \le 3 \Rightarrow n \le 2$



$$y[n] = \sum_{k=n+1}^{3} 9^k \cdot 3^{2n-2k}$$

$$= 3^{2n} \cdot \sum_{k=n+1}^{3} 1 = 9^n (3-n)$$

case 2 when $n+1 > 3 \Rightarrow n > 2$



$x[k]h[2n-2k] = 0$ for any k

$y[n] = 0$

Thus:

$$y[n] = \begin{cases} 9^n(3-n) & n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Alternative: $$\sum_{k=19}^{221} \sqrt{2} = \sqrt{2}\,(221 - 19 + 1)$$

$$= 203\sqrt{2}$$

Question 3: [15%, Work-out question] Consider two CT signals

$$x(t) = e^{jw \cdot t} \tag{6}$$

$$h(t) = \begin{cases} 2 & |t| \leq 1.5 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Find out the expression of $y(t) = \int_{-\infty}^{\infty} x(s)h(t-s)ds$.

Hint 1: It may be easier to apply some change of variables.

Hint 2: You can leave (part of) your answer to be of the form like $\frac{e^{3jt}-2^{-12t}}{3+10j}$. There is no need to further simplify the expression.

Answer :

$$x(s) = e^{jws}$$

$$h(t-s) = \begin{cases} 2 & |t-s| \leq 1.5 \Rightarrow t-1.5 \leq s \leq t+1.5 \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = \int_{s=t-1.5}^{t+1.5} 2e^{jws} ds$$

$$= \frac{2}{jw} e^{jws} \Big|_{s=t-1.5}^{t+1.5}$$

$$= \frac{4\cos(1.5w)}{jw} e^{jwt}$$

Question 4: [15%, Work-out question]

Consider the following DT signals.

$$x[n] = (3 + 4j)^{-n} \tag{8}$$
$$h[n] = e^{(1+2j)n} \cdot U[n - 50]. \tag{9}$$

Define

$$y[n] = x[n] \cdot h[n] \tag{10}$$

Find the *total energy* of $y[n]$.

Hint: If $|r| < 1$, then we have the following formulas for computing the infinite sum of a geometric sequence.

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$
$$\sum_{k=1}^{\infty} kar^{k-1} = \frac{a}{(1-r)^2}.$$

If $r \neq 1$, then we have the following formula for computing the finite sum of a geometric sequence.

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}.$$

Answer: $y[n] = \left(5e^{j\arctan\frac{4}{3}}\right)^{-n} e^{(1+2j)n} \, u[n-50]$

$$|y[n]| = \begin{cases} \left(\frac{e}{5}\right)^n & n \geq 50 \\ 0 & \text{otherwise} \end{cases} \quad (\text{because } |e^{j\theta}| = 1 \text{ for any } \theta)$$

$$E_{\text{Total}} = \sum_{n=-\infty}^{\infty} |y[n]|^2 = \sum_{n=50}^{\infty} \left(\frac{e}{5}\right)^{2n} = \frac{\left(\frac{e}{5}\right)^{100}}{1 - \left(\frac{e}{5}\right)^2}$$

Question 5: [15%, Work-out question]

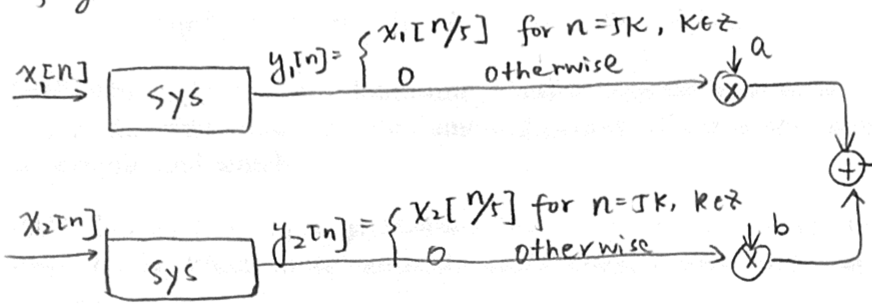
Consider the following DT system.

$$y[n] = \begin{cases} x[n/5] & \text{if } n \text{ is a multiple of 5, e.g., } n = -5, 0, 5, 10, \cdots \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

Is such a system *linear*? You need to carefully write down your reasonings. An answer without justification will not receive any point.

Answer :

Configuration 1 :



$$y_{Conf1}[n] = \begin{cases} a x_1[\frac{n}{5}] + b x_2[\frac{n}{5}] & \text{for } n = 5k,\ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Configuration 2 :

$$y_{Conf2}[n] = \begin{cases} a x_1[\frac{n}{5}] + b x_2[\frac{n}{5}] & \text{for } n = 5k,\ k = \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

$$y_{Conf1}[n] = y_{Conf2}[n]$$

The system is linear.

Question 6: [20%, Multiple Choices]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider two continuous-time signals:

$$x_1(t) = e^{j\cos 3t}\sin(7t - \pi) \tag{12}$$

$$x_2(t) = \sum_{k=2}^{10} \cos(3\sqrt{2k}\cdot t) \tag{13}$$

and two discrete-time signals:

$$x_3[n] = \sum_{k=0}^{|n|} \cos(\pi k^2) \tag{14}$$

$$x_4[n] = (U[n+4] - U[n-6] + \delta[n+5])\cdot \sin(0.25\pi n). \tag{15}$$

1. [10%] For $x_1(t)$ to $x_4[n]$, determine whether it is periodic or not, *respectively.* If it is periodic, write down the fundamental period. Please state explicitly which signal is periodic and which is not.

2. [10%] For $x_1(t)$ to $x_4[n]$, determine whether it is even or odd or neither of them, respectively. Please state explicitly which signal is even, which is odd, and which is neither.

Answer:

1. $x_1(t)$ is periodic with $T = 2\pi$.

$e^{j\cos 3t}$ is periodic with $T_1 = \frac{2\pi}{3}$

$\sin(7t-\pi)$ is periodic with $T_2 = \frac{2\pi}{7}$

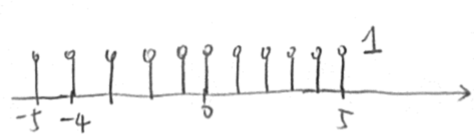
$T = LCM(T_1, T_2) = 2\pi$

$x_2(t)$ is aperiodic. $\sqrt{2k}$ is irrational.

$x_3[n]$ is periodic with $N = 2$.

$\cos(\pi k^2) = \begin{cases} 1 & k \text{ is even} \\ -1 & k \text{ is odd.} \end{cases} \Rightarrow x_3[n] = \begin{cases} 1 & k \text{ is even} \\ 0 & k \text{ is odd.} \end{cases}$

$x_4[n]$ is aperiodic

$x_4[n] =$  $\cdot \sin(\frac{\pi}{4}n)$

2. $x_1(t)$ is odd.

$e^{j\cos 3t}$ is even

$\sin(7t-\pi) = -\sin(7t)$ is odd

$x_2(t)$ is even

$$x_2(-t) = \sum_{k=2}^{10} \cos(-3\sqrt{2k}\,t) = \sum_{k=2}^{10} \cos(3\sqrt{2k}\,t) = x_2(t)$$

$x_3[n]$ is even

$x_4[n]$ is odd.