

**Final Exam of ECE 301-003 and OL1, (CRN: 11474 and 26295)
10:30am–12:30pm, Thursday, May 6, 2021, Online Exam.**

1. Enter your student ID number, and signature in the space provided on this page **now!**
2. This is a closed book exam.
3. This exam contains multiple-choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have two hours to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature:

Date:

Question 1: [23%, Work-out question]

1. [1%] What does the acronym AM-DSB stand for?
2. [1%] What does the acronym FDM stand for?

Prof. Wang wanted to transmit an AM-SSB signal. To that end, he wrote the following MATLAB code.

```
% Initialization
duration=8;
f_sample=44100;
t=((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5)/f_sample;

% Read two different .wav files
[x1, f_sample, N]=audioread('x1.wav');
x1=x1';
[x2, f_sample, N]=audioread('x2.wav');
x2=x2';

% Step 0: Initialize several parameters
W_1=pi*3500;
W_2=pi*2000;
W_3=????;
W_4=????;
W_5=????;
W_6=????;
W_7=????;

% Step 1: Make the signals band-limited.
h=1/(pi*t).*(sin(W_1*t));
x1_new=ece301conv(x1, h);
x2_new=ece301conv(x2, h);

% Step 2: Multiply x1_new and x2_new with a sinusoidal wave.
x1_h=x1_new.*sin(W_2*t);
x2_h=x2_new.*sin(W_3*t);

% Step 3: Keep one of the two side bands
h_one=1/(pi*t).*(sin(W_4*t)-sin(W_5*t));
h_two=1/(pi*t).*(sin(W_6*t).*(2*cos(W_7*t)));
x1_sb=ece301conv(x1_h, h_one);
x2_sb=ece301conv(x2_h, h_two);
```

```
% Step 4: Create the transmitted signal
y=x1_sb+x2_sb;
audiowrite('y.wav', y, f_sample);
```

3. [1%] What is the carrier frequency (Hz) of the signal $x1_new$?
4. [3%] Our goal is to transmit the “upper-side bands” for both $x1$ and $x2$ signals. What should be the values of W_4 and W_5 in the MATLAB code?
5. [3%] Suppose I choose the value of W_3 to 7000π , what should be the values of W_6 and W_7 in the MATLAB code?
6. [2%] What would be the smallest possible value of W_3 in the MATLAB code without negatively impacting the sound quality of either $x1$ and $x2$ signals? (Here we assume that we still use the upper side band and also the W_6 and W_7 values are adjusted accordingly when we choose the smallest possible W_3 value.)
7. [1.5%] If I choose the value of W_3 to be 5000π , please describe in words what would be the impact to the sound quality?

Prof. Wang decided to use the upper-side-band transmission with W_3 set to 7000π and used the code in the previous page to generate the “y.wav” file. A student tried to demodulate the output waveform “y.wav” by the following code.

```
% Initialization
duration=8;
f_sample=44100;
t=((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5)/f_sample;

% Read the .wav files
[y, f_sample, N]=audioread('y.wav');
y=y';

% Initialize several parameters
W_8=????;
W_9=????;
W_10=????;

% Create the low-pass filter.
h_M=1/(pi*t).*(sin(W_8*t));

% demodulate signal 1
y1=y.*sin(W_9*t);
x1_hat=ece301conv(y1,h_M);

sound(x1_hat,f_sample)

% demodulate signal 2
y2=y.*sin(W_10*t);
x2_hat=ece301conv(y2,h_M);

sound(x2_hat,f_sample)
```

8. [4.5%] Continue from the previous questions. What should the values of W_8 to W_{10} be in the MATLAB code?
9. [6%] It turns out that the above MATLAB code is not written correctly and the end results do not sound right. Neither signal $x1_new$ nor signal $x2_new$ can be correctly/perfectly demodulated. Please use 2 to 3 sentences to answer (i) what kind of problem does $x1_new$ have, i.e., how does the problem impact the sound quality of “sound($x1_hat, f_sample$)”? (ii) how can the MATLAB code be corrected so that the playback/demodulation can be performed successfully?

10. Please use 2 to 3 sentences to answer (iii) what kind of problem does `x2_new` have, i.e., how does the problem impact the sound quality of `“sound(x2_hat,f_sample)”`? (ii) how can the MATLAB code be corrected so that the playback/demodulation can be performed successfully? Please explain your changes very carefully for this question.

Sub-questions 9 and 10 would be graded together. The two sub-questions combined will have 6 points.

Hint: If you do not know the answers of Q1.3 to Q1.9, please simply draw the AMSSB modulation (using lower side band) and demodulation diagrams and mark carefully all the parameter values. You will receive 12 points for Q1.3 to Q1.9 if your system diagrams are correct and all parameter values are marked correctly.

Question 2: [14%, Work-out question]

1. [3%] Consider a continuous time signal $x(t)$

$$x(t) = \frac{\sin(3\pi t)}{\pi t}. \quad (1)$$

Plot $x(t)$ for the range of $-1 \leq t \leq 1$. Please carefully mark all the important points in the figure.

2. [3%] We sample $x(t)$ by an *impulse train sampler* with sampling frequency 4Hz. That is,

$$x_p(t) = x(t) \cdot \left(\sum_{k=-\infty}^{\infty} \delta(t - 0.25k) \right) \quad (2)$$

Plot $x_p(t)$ for the range of $-0.4 \leq t \leq 0.4$. Please carefully mark all the important points in the figure.

Hint: $\sin(0.25\pi) = \sin(0.75\pi) = \frac{\sqrt{2}}{2}$.

3. [4%] We know that the frequency domain $X(j\omega)$ of the signal $x(t)$ is

$$X(j\omega) = \begin{cases} 1 & \text{if } -3\pi \leq \omega \leq 3\pi \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Plot the frequency domain $X_p(j\omega)$ of $x_p(t)$ for the range of $-12\pi \leq \omega \leq 12\pi$.

4. [4%] We then pass $x_p(t)$ over a *Band-Pass Filter* with the following frequency response

$$h_{\text{BPF}}(t) = \frac{1}{8} \cdot \frac{\sin((8000\pi + 4\pi) \cdot t) - \sin((8000\pi - 4\pi) \cdot t)}{\pi t}. \quad (4)$$

That is, $y(t) = x_p(t) * h_{\text{BPF}}(t)$. Plot $y(t)$ for the range of $-1 \leq t \leq 1$.

Hint 1: There is a connection between this question and the concept of AM-DSB. Such a connection will be critical when solving this question.

Hint 2: If you do not know how to solve Q2.4, you can write down the description of AM *asynchronous demodulation*. You will receive 1.5 points if your answer is correct.

Hint 3: This sub-question is considered to be slightly more advanced. You may consider working on other questions first and then come back to this one later.

Question 3: [12%, Work-out question]

1. [3%] Consider the following continuous time signal

$$x(t) = \cos(5\pi t) \quad (5)$$

We sample $x(t)$ with the sampling frequency 20Hz and denote the sampled values by $x[n]$. Plot $x[n]$ for the range of $-3 \leq n \leq 3$.

Hint: $\cos(0.25\pi) = \frac{\sqrt{2}}{2}$.

2. [3%] We use $x_{\text{ZOH}}(t)$ to represent the reconstructed signal using “zero-order hold”. Plot $x_{\text{ZOH}}(t)$ for the range of $-0.15 \leq t \leq 0.15$.

Hint: if you do not know the answer of $x[n]$, you can assume that $x[n] = 2(-1)^{n+1} - 2$ and the sampling frequency is 20Hz. You will receive full points if your answer is correct.

3. [3%] We use $x_{\text{opt}}(t)$ to represent the optimal band-limited reconstruction of $x(t)$. Plot $x_{\text{opt}}(t)$ for the range of $-0.4 \leq t \leq 0.4$.

Hint 1: if you do not know the answer of this sub-question, please write down the description of “sampling theorem”. You will receive 1.5 points if your answer is correct.

Hint 2: Sampling theorem may be useful when answering this question.

4. [3%] Suppose we pass the sampled array $x[n]$ through a DT-LTI system with impulse response $h[n] = \delta[n - 4]$ and denote the output array to be $y[n]$. We use $y(t)$ to represent the optimal band-limited reconstruction based on the output array $y[n]$ and the fact that the sampling frequency is 20Hz. Plot $y(t)$ for the range of $-0.4 \leq t \leq 0.4$.

Hint 3: This sub-question is considered to be slightly more advanced and is only worth 3 points. You may consider working on other questions first and then come back to this one later.

Question 4: [9%, Work-out question]

1. [2.5%] Consider a discrete time $x[n]$ and we know its DTFT $X(e^{j\omega})$ inside the range of $-\pi \leq \omega \leq \pi$ being

$$X(e^{j\omega}) = \pi\delta(\omega - 0.1\pi) + \pi\delta(\omega + 0.1\pi) \quad \text{if } -\pi \leq \omega \leq \pi \quad (6)$$

Plot $X(e^{j\omega})$ for the range of $-3\pi \leq \omega \leq 3\pi$.

2. [2.5%] Find the expression of $x[n]$.
3. [4%] Define $Y(e^{j\omega})$ as

$$Y(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) *_{\text{cyc.}} \left(\frac{2\pi}{4} \sum_{k=-\infty}^{\infty} \delta(\omega - k \cdot \frac{2\pi}{4}) \right) \quad (7)$$

where “ $*_{\text{cyc.}}$ ” represents the *cyclic convolution*. Find (i) the expression of $y[n]$; (ii) the value of $y[2]$; and (iii) the value of $y[4]$. The three answers will be graded together.

Hint 1: The DTFT table would be very useful when solving this question. Specifically, the table can help you find the relationship between $y[n]$ and $x[n]$.

Hint 2: If you do not know the answer to this sub-question, please write down the definition of *cyclic convolution*. You will receive 2 points if your answer is correct.

Question 5: [8%, Work-out question]

Consider the following continuous time signals

$$x(t) = \sum_{k=1}^2 e^{j \cdot (kt - \sqrt{k})} \quad (8)$$

$$h(t) = e^{-|t-3|} \quad (9)$$

Find the expression of $y(t) = x(t) * h(t)$.

Hint: For a signal $w(t) = e^{-|t|}$, we know its Fourier transform being $W(j\omega) = \frac{2}{1+\omega^2}$.
You may use this fact plus the time-shift property to simplify your computation.

Question 6: [9%, Work-out question]

Consider the following discrete-time signal

$$x[n] = \begin{cases} \frac{n+2}{3} & \text{if } -2 \leq n \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

Denote the corresponding DTFT by $X(e^{j\omega})$.

1. [4%] Find the value of $\int_{\omega=-\pi}^{\pi} X(e^{j\omega})e^{j\omega}d\omega$.
2. [5%] Find the value of $\int_{\omega=-\pi}^{\pi} |X(e^{j\omega})|^2d\omega$.

Question 7: [10%, Work-out question]

Consider the following discrete time signal:

$$x[n] = \begin{cases} (3j)^{n+1} & \text{if } n \leq -1 \\ 0 & \text{if } n = 0 \\ 0.5^{n-1} & \text{if } 1 \leq n \end{cases} \quad (11)$$

- Derive the Z-transform expression $X(z)$ of $x[n]$;
- and plot the corresponding region of convergence.

These two questions will be graded together since it is testing your knowledge about how to derive the Z-transform. Please carefully write down your reasonings. If you use the table without explanation, then you will only receive 7 points even if your answer is correct.

Hint 1: You may need the following formula:

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \quad (12)$$

Hint 2: If you do not know the answer to any of the above two question, please answer the following yes/no question: Suppose a signal $x[n]$ has its Z-transform being $\frac{z^7}{1-2jz^{-1}}$ with the ROC being $|z| > 2$. Does the DTFT of $x[n]$ exist? Please carefully write down your reasons. If your reasons are correct, you will receive 4 points. A yes/no answer without justification will receive no point even if it may be correct.

Question 8: [15%, Multiple-choice question] Consider two signals

$$h_1(t) = \begin{cases} \left(\frac{e^{4jt} - e^{-4jt}}{2j\pi t} \right)^2 & \text{if } t \neq 0 \\ \frac{16}{\pi^2} & \text{if } t = 0 \end{cases} \quad (13)$$

and

$$h_2[n] = (\cos(n) + \cos(|n|)) \cdot (\sin(n) + \sin(|n|)) \quad (14)$$

1. [1.25%] Is $h_1(t)$ periodic?
2. [1.25%] Is $h_2[n]$ periodic?
3. [1.25%] Is $h_1(t)$ even or odd or neither?
4. [1.25%] Is $h_2[n]$ even or odd or neither?
5. [1.25%] Is $h_1(t)$ of finite power?
6. [1.25%] Is $h_2[n]$ of finite power?

Suppose the above two signals are also the impulse responses of two LTI systems: System 1 and System 2, respectively.

1. [1.25%] Is System 1 memoryless?
2. [1.25%] Is System 2 memoryless?
3. [1.25%] Is System 1 causal?
4. [1.25%] Is System 2 causal?
5. [1.25%] Is System 1 stable?
6. [1.25%] Is System 2 stable?

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \quad (1)$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \quad (2)$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \quad (3)$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \quad (4)$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad (5)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad (6)$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad (7)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad (8)$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \quad (9)$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad (10)$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \quad (11)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad (12)$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$ } Periodic with period T and $y(t)$ } fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a_{-k}^*
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t - \tau)d\tau$	$T a_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk \frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^t x(t) dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a_{-k}^* \\ \Re\{a_k\} = \Re\{a_{-k}\} \\ \Im\{a_k\} = -\Im\{a_{-k}\} \\ a_k = a_{-k} \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\begin{cases} \Re\{a_k\} \\ \Im\{a_k\} \end{cases}$
Parseval's Relation for Periodic Signals			
$\frac{1}{T} \int_T x(t) ^2 dt = \sum_{k=-\infty}^{+\infty} a_k ^2$			

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t - 1) - 1/2. \quad (3.40)$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$x[n]$ } Periodic with period N and $y[n]$ } fundamental frequency $\omega_0 = 2\pi/N$	a_k } Periodic with b_k } period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a_{-k}^*
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k$ (viewed as periodic) (with period mN)
Periodic Convolution	$\sum_{r=(N)} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=(N)} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^n x[k]$ (finite valued and periodic only) (if $a_0 = 0$)	$\left(\frac{1}{1 - e^{-jk(2\pi/N)}}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a_{-k}^* \\ \Re\{a_k\} = \Re\{a_{-k}\} \\ \Im\{a_k\} = -\Im\{a_{-k}\} \\ a_k = a_{-k} \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\begin{cases} \Re\{a_k\} \\ j\Im\{a_k\} \end{cases}$
Parseval's Relation for Periodic Signals		
$\frac{1}{N} \sum_{n=(N)} x[n] ^2 = \sum_{k=(N)} a_k ^2$		

4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
<hr/>			
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^t x(t)dt$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j \frac{d}{d\omega} X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re\{X(j\omega)\} = \Re\{X(-j\omega)\} \\ \Im\{X(j\omega)\} = -\Im\{X(-j\omega)\} \\ X(j\omega) = X(-j\omega) \\ \angle X(j\omega) = -\angle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}\{x(t)\}$ [$x(t)$ real]	$\Re\{X(j\omega)\}$ $j\Im\{X(j\omega)\}$
<hr/>			
4.3.7	Parseval's Relation for Aperiodic Signals		
		$\int_{-\infty}^{+\infty} x(t) ^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) ^2 d\omega$	

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi \delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi \delta(\omega)$	$a_0 = 1$, $a_k = 0$, $k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave		
$x(t) = \begin{cases} 1, & t < T_1 \\ 0, & T_1 < t \leq \frac{T}{2} \end{cases}$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
and $x(t + T) = x(t)$		
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & t < T_1 \\ 0, & t > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \omega < W \\ 0, & \omega > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi \delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at} u(t), \operatorname{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$t e^{-at} u(t), \operatorname{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t), \operatorname{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$	$X(e^{j\omega})$ periodic with
		$y[n]$	$Y(e^{j\omega})$ period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0} X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi} \int_{2\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n - 1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1 - e^{-j\omega}} X(e^{j\omega})$
5.3.8	Differentiation in Frequency	$nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \Re\{X(e^{j\omega})\} = \Re\{X(e^{-j\omega})\} \\ \Im\{X(e^{j\omega})\} = -\Im\{X(e^{-j\omega})\} \\ X(e^{j\omega}) = X(e^{-j\omega}) \\ \angle X(e^{j\omega}) = -\angle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real and even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}\{x[n]\}$ [$x[n]$ real] $x_o[n] = \mathcal{O}\{x[n]\}$ [$x[n]$ real]	$\Re\{X(e^{j\omega})\}$ $j\Im\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals	$\sum_{n=-\infty}^{+\infty} x[n] ^2 = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) ^2 d\omega$	

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

5.7.1 Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients a_k of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence a_k in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence a_k are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=(N)} a_k e^{jk(2n/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational \Rightarrow The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational \Rightarrow The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational \Rightarrow The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & n \leq N_1 \\ 0, & N_1 < n \leq N/2 \end{cases}$ and $x[n + N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}$, $k \neq 0, \pm N, \pm 2N, \dots$ $a_k = \frac{2N_1 + 1}{N}$, $k = 0, \pm N, \pm 2N, \dots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n]$, $ a < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & n \leq N_1 \\ 0, & n > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \text{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq \omega \leq W \\ 0, & W < \omega \leq \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n + 1)a^n u[n]$, $ a < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r - 1)!} a^n u[n]$, $ a < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

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Chap. 10

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R , except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$ Scaled version of R (i.e., $ a R =$ the set of points $\{ a z\}$ for z in R)
		$a^r x[n]$	$X(a^{-1}z)$	Inverted R (i.e., $R^{-1} =$ the set of points z^{-1} , where z is in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.5	Time expansion	$x_{(r)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$	$X(z^k)$	R At least the intersection of R_1 and R_2 At least the intersection of R and $ z > 0$ At least the intersection of R and $ z > 1$
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	R
10.5.7	First difference	$x[n] - x[n - 1]$	$(1 - z^{-1})X(z)$	R
10.5.7	Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	R
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z \frac{dX(z)}{dz}$	R
10.5.9			Initial Value Theorem If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \rightarrow \infty} X(z)$	

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$ z > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$ z < 1$
4. $\delta[n-m]$	z^{-m}	All z , except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$ z > \alpha $
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$ z < \alpha $
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$ z > \alpha $
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$ z < \alpha $
9. $[\cos \omega_0 n]u[n]$	$\frac{1 - [\cos \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$	$ z > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$	$ z > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1 - [r \cos \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$	$ z > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r \sin \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$	$ z > r$

10.7.1 Causality

A causal LTI system has an impulse response $h[n]$ that is zero for $n < 0$, and therefore is right-sided. From Property 4 in Section 10.2 we then know that the ROC of $H(z)$ is the exterior of a circle in the z -plane. For some systems, e.g., if $h[n] = \delta[n]$, so that $H(z) = 1$, the ROC can extend all the way in to and possibly include the origin. Also, in general, for a right-sided impulse response, the ROC may or may not include infinity. For example, if $h[n] = \delta[n+1]$, then $H(z) = z$, which has a pole at infinity. However, as we saw in Property 8 in Section 10.2, for a causal system the power series

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$$

does not include any positive powers of z . Consequently, the ROC includes infinity. Summarizing, we have the follow principle:

A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity.