Midterm #3 of ECE 301-004, (CRN: 17102)
6:30–7:30pm, Tuesday, November 16, 2021, FRNY G140.

1. **Do not write answers on the back of pages!**
2. **After the exam ended, you will have 5 additional minutes to write down your name and Purdue ID on each of the pages.**
3. **If you need additional sheets to write down your answers, please let the instructor/TA know. We will hand out additional answer sheets then.**
4. Enter your student ID number, and signature in the space provided on this page.
5. This is a closed book exam.
6. This exam contains multiple-choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.
7. The instructor/TA will hand out loose sheets of paper for the rough work.
8. Neither calculators nor help sheets are allowed.

Name: Answer

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

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Question 1: [22%, Work-out question]

Consider a continuous time periodic signal $x(t) = \sum_{k=2}^{3} \cos(\frac{k\pi}{4}t)$. We know the period of $x(t)$ is 8.

1. [6%] Denote the CTFS coefficients of $x(t)$ by a_k. Find the values of a_k for all k.

Consider an LTI system with impulse response being

$$h(t) = \begin{cases} 1 & \text{if } 0 \le t \le 4 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Note that the above $h(t)$ is an *off-center* rectangular waveform.

2. [8%] Find the frequency response $H(j\omega)$.

3. [8%] When the input is $x(t)$ described in Q1.1, let $y(t)$ denote the corresponding output of the above LTI system. Let b_k denote the corresponding CTFS coefficients of $y(t)$. Find the values of b_2 and b_3, respectively.

   Hint 1: If there is something like $\sin(\pi/2)$ in your answer, you need to simplify it by using $\sin(\pi/2) = 1$. Namely, the following sinusoidal function values

   $$\sin(0) = \sin(\pi) = 0; \quad \sin(0.5\pi) = 1; \quad \sin(1.5\pi) = -1 \tag{2}$$

   are needed when simplifying your answer. In your end, your answer would be something like $e^{j0.33\pi} \cdot \frac{\pi}{3}$.

   Hint 1: If you do not know the answers to Q1.1 and Q1.2, you can write down your answer for b_2 and b_3 as a function of a_k and $H(j\omega)$. You will still receive 4 points if your answer is correct.

   Hint 2:

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This sheet is for Question 1.

Answer:

1.1) $x(t) = \cos\left(\frac{2\pi}{4}t\right) + \cos\left(\frac{3\pi}{4}t\right)$ with $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{8} = \frac{\pi}{4}$

$= \frac{1}{2}e^{j\frac{2\pi}{4}t} + \frac{1}{2}e^{-j\frac{2\pi}{4}t} + \frac{1}{2}e^{j\frac{3\pi}{4}t} + \frac{1}{2}e^{-j\frac{3\pi}{4}t}$

$\therefore$ $a_{-3} = \frac{1}{2}$, $a_{-2} = \frac{1}{2}$, $a_2 = \frac{1}{2}$ and $a_3 = \frac{1}{2}$

$a_k = 0$ for other $k \in \mathbb{Z}$

1.2) $H(jw) = \int_{-\infty}^{+\infty} h(t)e^{-jwt}\,dt$

$= \int_{t=0}^{4} e^{-jwt}\,dt = \left.\frac{e^{-jwt}}{-jw}\right|_{t=0}^{4}$

$= \frac{1}{-jw}\left(e^{-4jw} - 1\right) = 2e^{-2jw} \cdot \frac{\sin 2w}{w}$

1.3) $b_2 = H\left(j\frac{2\pi}{4}\right) \cdot a_2 = 0$ (because $\sin\left(2 \cdot \frac{\pi}{2}\right) = 0$)

$b_3 = H\left(j\frac{3\pi}{4}\right) \cdot a_3 = e^{-j\frac{3\pi}{2}} \cdot \frac{\sin\frac{3\pi}{2}}{\frac{3\pi}{4}} = -\frac{4}{3\pi}j$.

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Question 2: [16%, Work-out question] Consider a CT signal

$$x(t) = \frac{\sin(t)\sin(2t)\sin(8t)}{\pi^2 t^2} \tag{3}$$

Find its Fourier transform $X(j\omega)$.

Hint: If you do not know how to solve the above question, you can solve the following alternative question: We know $y(t) = \frac{\sin(t)\sin(2t)}{\pi^2 t^2}$, find its Fourier transform $Y(j\omega)$. You will receive 12 points if your answer is correct.

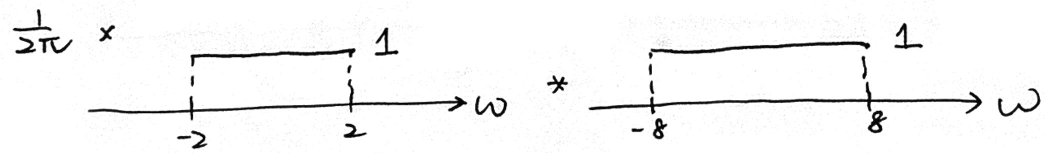
Answer :

$$X(t) = \sin(t) \cdot \frac{\sin(2t)}{\pi t} \cdot \frac{\sin(8t)}{\pi t}$$

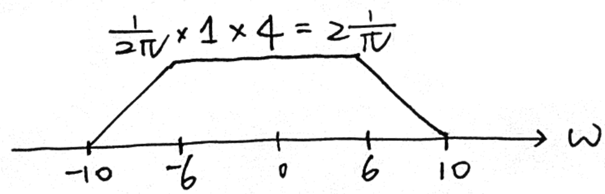
By the multiplication property

$$X_1(t) = \frac{\sin(2t)}{\pi t} \cdot \frac{\sin(8t)}{\pi t}$$

$$\longleftrightarrow \quad X_1(jw) = \frac{1}{2\pi} \mathcal{F}\left\{\frac{\sin(2t)}{\pi t}\right\} * \mathcal{F}\left\{\frac{\sin(8t)}{\pi t}\right\}$$



$\Rightarrow$



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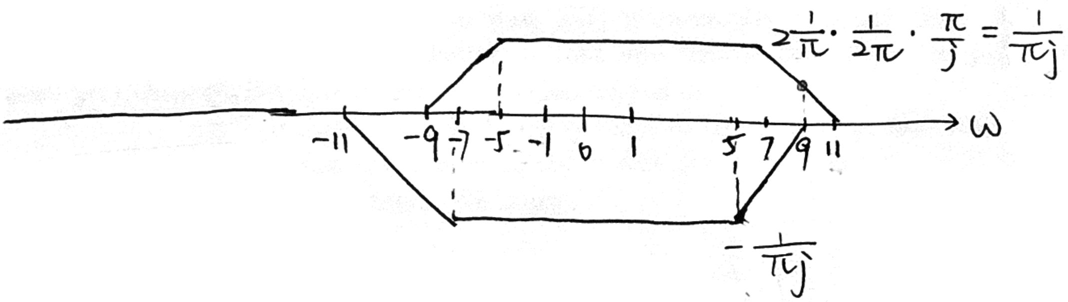
This sheet is for Question 2.

$$x(t) = \sin(t) \cdot x_1(t)$$

$$\longleftrightarrow \quad X(j\omega) = \frac{1}{2\pi} \mathcal{F}\{\sin(t)\} * X_1(j\omega)$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{j}\delta(\omega-1) - \frac{\pi}{j}\delta(\omega+1)\right) * X_1(j\omega)$$



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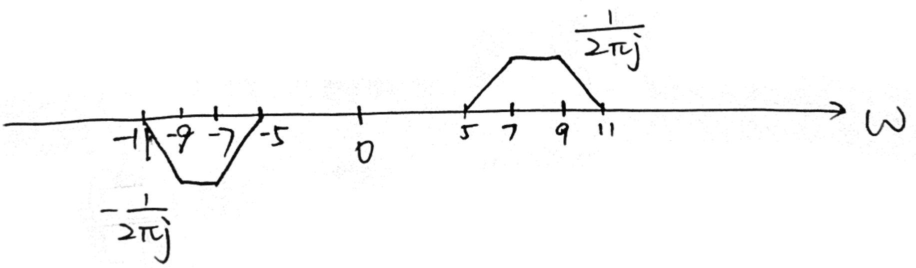

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Question 3: [14%, Work-out question] Consider two LTI systems, System 1 and System 2. System 1 has an impulse response

$$h_1(t) = e^{-t}\mathcal{U}(t) \tag{4}$$

and System 2 has an impulse response

$$h_2(t) = e^{-2t}\mathcal{U}(t) \tag{5}$$

Consider an input signal $x(t)$ being

$$x(t) = e^{-j3t}\mathcal{U}(t) \tag{6}$$

Find the output $z(t)$ when we pass $x(t)$ sequentially through these two systems. Namely, we pass $x(t)$ through System 1 first and denote the output as $y(t)$. Then we pass $y(t)$ through System 2 and denote the output by $z(t)$.

Hint 1: It may be easier to convert everything to the frequency domain.

Hint 2: If you do not know how to solve this question, you can solve the following *partial fraction* question instead: We know

$$\frac{1}{(j\omega+2)(j\omega+3)} = \frac{a}{j\omega+2} + \frac{b}{j\omega+3}. \tag{7}$$

Find the values of a and b. If your answer is correct, you will receive 9 points.

Answer: Let $h(t) = h_1(t) * h_2(t)$

$$H(j\omega) = \frac{1}{1+j\omega} \cdot \frac{1}{2+j\omega}$$

$$= \frac{a}{1+j\omega} + \frac{b}{2+j\omega} \quad \left(\begin{cases} a+b=0 \\ 2a+b=1 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=-1 \end{cases} \right)$$

$$= \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

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This sheet is for Question 3.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow z(t)$$

$$z(t) = x(t) * h(t)$$

$$\longleftrightarrow \quad Z(j\omega) = X(j\omega) \cdot H(j\omega)$$

$$= \frac{1}{3+j\omega} \cdot \frac{1}{1+j\omega} - \frac{1}{3+j\omega} \cdot \frac{1}{2+j\omega}$$

$$= \frac{-\frac{1}{2}}{3+j\omega} + \frac{\frac{1}{2}}{1+j\omega} + \frac{1}{3+j\omega} - \frac{1}{2+j\omega}$$

$$= \frac{\frac{1}{2}}{3+j\omega} + \frac{\frac{1}{2}}{1+j\omega} - \frac{1}{2+j\omega}$$

$$\longleftrightarrow \quad z(t) = \frac{1}{2} e^{-3t} u(t) + \frac{1}{2} e^{-t} u(t) - e^{-2t} u(t)$$

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Question 4: [21%, Work-out question] Consider the following periodic CT signal $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k\pi) - \delta(t - (2k+1)\pi) \tag{8}$$

1. [4%] Plot $x(t)$ for the range of $-1.5\pi < t < 4.5\pi$.
2. [14%] Find the expression of the Fourier transform $X(j\omega)$.

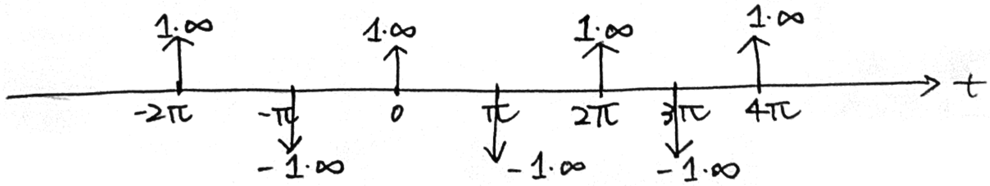
   Hint 1: This type of computation is termed the *generalized CTFT* in the lecture.

   Hint 2: If you do not know how to find the CTFT of $x(t)$, you should find the CTFS of $x(t)$ instead. You will receive 9 points out of your Q4.2 if your answer is correct.

   Hint 3: The CTFS of $x(t)$ actually helps you find the CTFT of $x(t)$.
3. [3%] Plot $X(j\omega)$ for the range of $-3.5 < \omega < 3.5$.

Answer:

4.1) $x(t)$



4.2) $x(t) = \sum_{k=-\infty}^{+\infty} \delta(t-2k\pi) - \sum_{k=-\infty}^{+\infty} \delta(t-(2k+1)\pi)$

$T_1 = 2\pi$ $\quad$ $T_2 = 2\pi$ with shifting $t_0 = \pi$

$$\longleftrightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} \delta(\omega - k) - e^{-j\omega\pi} \sum_{k=-\infty}^{+\infty} \delta(\omega - k)$$

$$= \sum_{k=-\infty}^{+\infty} \delta(\omega - k) - \sum_{k=-\infty}^{+\infty} e^{-jk\pi} \delta(\omega - k)$$

$$= \sum_{k=-\infty}^{+\infty} (1 - e^{-jk\pi}) \delta(\omega - k) = \sum_{\ell=-\infty}^{+\infty} 2\delta(\omega - 1 - 2\ell)$$

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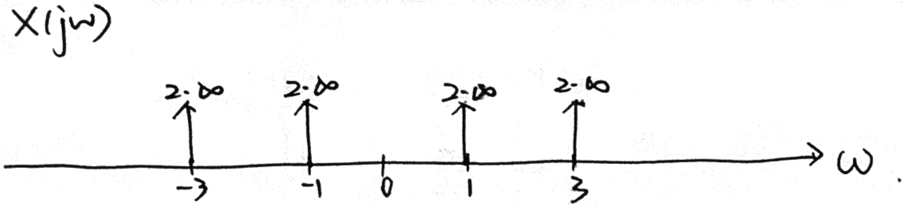
4.3)



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Question 5: [12%, Work-out question] Consider a discrete time signal $x[n]$:

$$x[n] = \begin{cases} n & \text{if } -1 \leq n \leq 1 \\ 2^{-(n-1)} & \text{if } 2 \leq n \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Find its Fourier transform $X(e^{j\omega})$.
Hint 1: The following formulas may be useful: If $|r| < 1$, then

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}. \tag{10}$$

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a \cdot (1-r^K)}{1-r}. \tag{11}$$

Hint 2: Your answer would be something like the following: $\cos(2\omega) + \frac{5e^{j6\omega}}{3+4e^{j\omega}}$. There is no need to further simplify it.

Answer: $X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$

$= 0 + (-1)\cdot e^{jw} + 0 + (1)\cdot e^{-jw} + \sum_{n=2}^{+\infty} 2^{-(n-1)} e^{-jwn}$

$= -e^{jw} + e^{-jw} + 2 \cdot \sum_{n=2}^{+\infty} \left(\frac{1}{2} e^{-jw}\right)^n$

$= -e^{jw} + e^{-jw} + 2 \cdot \frac{\frac{1}{4} e^{-j2w}}{1 - \frac{1}{2} e^{-jw}}$

$= -2j\sin w + \frac{1}{2} \frac{e^{-j2w}}{1 - \frac{1}{2} e^{-jw}}$

Question 6: [15%, Work-out question] Consider the following AM transmission system. The input signal is $x(t) = \frac{\sin(2\pi t)}{\pi t}$. We first multiply $x(t)$ by $\cos(4\pi t)$. That is,

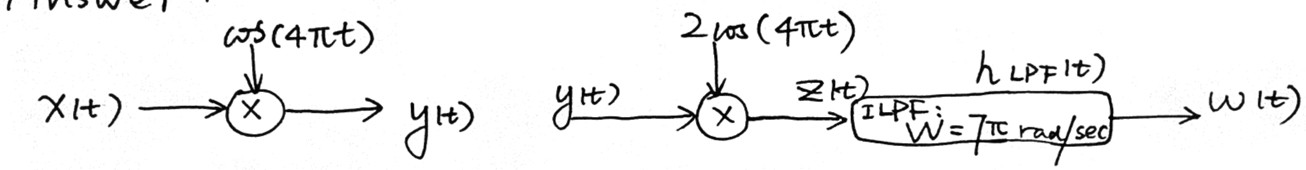
$$y(t) = x(t) \cdot \cos(4\pi t).$$

The transmitter then transmits signal $y(t)$ through the antenna.

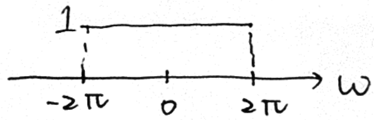
At the receiver side, we first multiply $y(t)$ by $2\cos(4\pi t)$. That is $z(t) = y(t) \cdot 2\cos(4\pi t)$ and then pass $z(t)$ through a low pass filter with cutoff frequency $W = 7\pi$ rad/sec. Denote the final output by $w(t) = z(t) * h_{\text{LPF}}(t)$.

1. [5%] Plot the CTFT $Y(j\omega)$ of $y(t)$ for the range of $-11\pi < \omega < 11\pi$.
2. [5%] Plot the CTFT $Z(j\omega)$ of $z(t)$ for the range of $-11\pi < \omega < 11\pi$.
3. [5%] Plot the CTFT $W(j\omega)$ of $w(t)$ for the range of $-11\pi < \omega < 11\pi$.

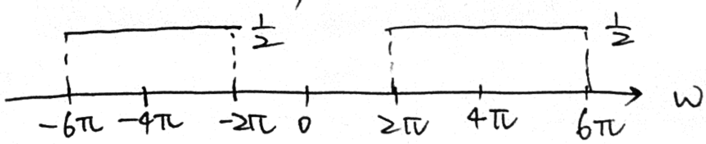
Answer:



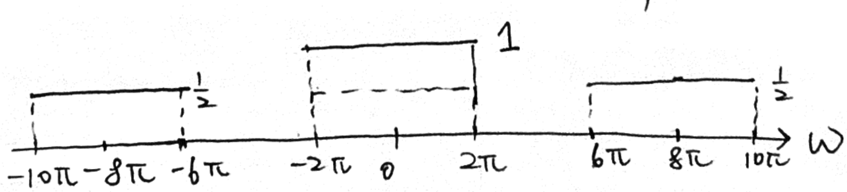
6.1) $x(t) \xleftrightarrow{FT} X(j\omega)$



$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * \mathcal{F}\{\cos(4\pi t)\}$



6.2) $Z(j\omega) = \frac{1}{2\pi} Y(j\omega) * \mathcal{F}\{2\cos(4\pi t)\}$



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6.3) $W(j\omega)$