Midterm #2 of ECE 301-004, (CRN: 17102)
8–9pm, Tuesday, October 19, 2021, FRNY G140.

1. Enter your student ID number, and signature in the space provided on this page **now!**
2. This is a closed book exam.
3. This exam contains multiple-choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.
4. The instructor/TA will hand out loose sheets of paper for the rough work.
5. **If you need additional sheets to write down your answers, please let the instructor/TA know. We will hand out additional answer sheets then.**
6. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date: 2021/10/18

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Question 1: [20%, Work-out question]

Consider a DT-LTI system for which we know that if the input is

$$x_1[n] = \begin{cases} 2 & \text{if } n = 1 \\ 0 & \text{if } n \neq 0 \end{cases} \tag{1}$$

then the output is

$$y_1[n] = \begin{cases} 2 & \text{if } 1 \leq n \\ 2e^{n-1} & \text{if } n \leq 0 \end{cases} \tag{2}$$

1. [6%] Find the *impulse response* $h[n]$ of this system.

   Hint 1: If you do not know the answer to this question, please write down (i) What is the definition of *impulse response*; (ii) what does "LTI" stand for? If your answers are correct, you will receive 3 points; (iii) Plot $x_1[n]$ for the range of $-2 \leq n \leq 2$.

   Hint 2: Your answer to (i), (ii), and (iii) in the above hint plays an important role when solving Q1.1 as well.

2. [14%] If the input to this DT-LTI system is $x_2[n] = e^{-n}U[n]$, find the corresponding output $y_2[n]$.

   Hint 3: If you do not know the answer of the impulse response in Q1.1, you can assume $h[n] = y_1[n]$ (even though it is not true) and use $h[n] = y_1[n]$ when solving this question. You will receive full credit if your answer (under this false assumption) is correct.

   Hint 4: You may need the following formulas:

$$\text{if } |r| < 1, \text{ then } \sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \tag{3}$$

$$\text{if } r \neq 1, \text{ then } \sum_{k=1}^{K} a \cdot r^{k-1} = \frac{a \cdot (1 - r^K)}{1-r} \tag{4}$$

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This sheet is for Question 1.

Answer:

Q1.1 $x_1[n] = 2\delta[n-1] \rightarrow \boxed{\text{sys}} \rightarrow y_1[n]$

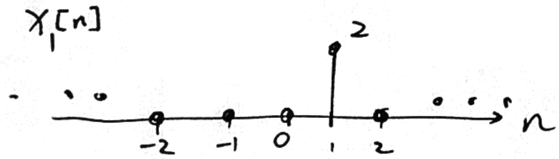
Because the system is time-invariant, and linear

$$x[n] = \delta[n] \rightarrow \boxed{\text{sys}} \rightarrow y[n] = \frac{y_1[n+1]}{2}$$

The definition of impulse response:

The output of a LTI system when the input is $\delta[n]$.

"LTI" stands for linear and time-invariant.



$$h[n] = y[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ e^n & \text{if } n \leq -1 \end{cases}$$

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This sheet is for Question 1.

Q1.2 $x_2[n] \to \boxed{\text{sys}} \to y_2[n]$

$$y_2[n] = x_2[n] * h[n]$$

$$= \sum_{k=-\infty}^{+\infty} x_2[n-k]\,h[k]$$

when $n \le -1$

$$y_2[n] = \sum_{k=-\infty}^{n} e^{-(n-k)} e^{k}$$

$$= e^{-n} \sum_{k=-\infty}^{n} e^{2k} = e^{-n} \frac{e^{+2n}}{1-e^{-2}}$$

$$= \frac{e^{n}}{1-e^{-2}}$$

when $n > -1$

$$y_2[n] = \sum_{k=-\infty}^{-1} e^{-(n-k)} e^{k} + \sum_{k=0}^{n} e^{-(n-k)}$$

$$= e^{-n} \frac{e^{-2}}{1-e^{-2}} + e^{-n} \frac{1(1-e^{n+1})}{1-e}$$

$$= \frac{e^{-(n+2)}}{1-e^{-2}} + \frac{e^{-n}-e}{1-e}$$

Hint 3: $x_3[n] \to \boxed{y_1[n]} \to y_2[n]$

$$y_2[n] = 2 \cdot \frac{e^{-(n+1)}}{1-e^{-2}} + 2 \cdot \frac{e^{-n+1}-e}{1-e}$$

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Question 2: [14%, Work-out question] Consider a CT system with the following input/output relationship:

$$y(t) = \int_{-4}^{2} x(t-s)e^{-s}e^{js}ds. \tag{5}$$

Find the expression of the impulse response $h(t)$.

Answer:

$$y(t) = \int_{s=-4}^{2} x(t-s)e^{-s}e^{js}\,ds$$

Chang variable, let $\tau = t-s \Rightarrow s = t-\tau$

$$y(t) = -\int_{t-\tau=-4}^{2} x(\tau)e^{-(t-\tau)}e^{j(t-\tau)}\,d\tau$$

$$= \int_{\tau=t-2}^{t+4} x(\tau)e^{(-1+j)(t-\tau)}\,d\tau$$

$$h(t) = e^{(-1+j)t}\left[u(t+4) - u(t-2)\right]$$

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Question 3: [14%, Work-out question]

Consider a CT-LTI system with the impulse response being $h(t) = e^{(-j+5)t}U(-t)$. Let $y(t)$ denote the output when the input is $x(t) = e^{j4t}$. Find the absolute value of $y(3)$, i.e., find $|y(3)|$, and also find the angle of $y(3)$, i.e., find $\angle y(3)$.

Answer :

$$y(t) = h(t) * x(t)$$

$$= \int_{s=-\infty}^{+\infty} x(s)\, h(t-s)\, ds$$

$$= \int_{t}^{+\infty} e^{j4s} e^{(-j+5)(t-s)} ds$$

$$= e^{(-j+5)t} \int_{t}^{+\infty} e^{(-5+5j)s} ds$$

$$= e^{(5-j)t} \left. \frac{e^{(-5+5j)s}}{-5+5j} \right|_{s=t}^{+\infty}$$

$$= e^{(5-j)t} \left(0 - \frac{e^{(-5+5j)t}}{-5+5j} \right)$$

$$= \frac{e^{4jt}}{5-5j} = \frac{1}{5\sqrt{2}} \frac{e^{4jt}}{e^{-\frac{\pi}{4}j}}$$

$$= \frac{1}{5\sqrt{2}} e^{(4t+\frac{\pi}{4})j}$$

$$|y(3)| = \frac{1}{5\sqrt{2}} \qquad \measuredangle y(3) = 12 + \frac{\pi}{4} = 12 - \frac{7\pi}{4}$$

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Question 4: [18%, Work-out question] Consider a periodic CT signals

$$x(t) = \begin{cases} -2+j & \text{if } 5 \le t < 25 \\ 0-2j & \text{if } 25 \le t < 35 \\ 1 & \text{if } 35 \le t < 45 \\ \text{periodic with period } T = 40 \end{cases} \quad (6)$$

We denote the Fourier series of $x(t)$ by (a_k, ω_0) where $\omega_0 = \frac{2\pi}{40}$.

1. [6%] Find the value of a_0.
2. [6%] Find the value of $\sum_{k=-\infty}^{\infty} a_k$.
   Hint: We know that the CTFS synthesis formula is $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$.
3. [6%] Find the value of $\sum_{k=-\infty}^{\infty} |a_k|^2$.

Answer:

4.1. $a_k = \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}\,d\tau$

$$a_0 = \frac{1}{40}\int_5^{45} x(\tau)\,d\tau$$

$$= \frac{1}{40}\left\{(-2+j)\times(25-5) + (-2j)(35-25) + 1\times(45-35)\right\}$$

$$= \frac{1}{40}\cdot(-30) = -\frac{3}{4}$$

4.2. By the synthesis equation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

When $t=0$, $x(0) = \sum_{k=-\infty}^{+\infty} a_k = x(40)$

$$= 1$$

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This sheet is for Question 4.

4.3. By Parseval's law

$$\frac{1}{T}\int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

$$|x(t)| = \begin{cases} \sqrt{4+1} = \sqrt{5} & 5 \le t < 25 \\ \sqrt{0+4} = 2 & 25 \le t < 35 \\ 1 & 35 \le t < 45 \end{cases}$$

$$\sum_{k=-\infty}^{+\infty} |a_k|^2 = \frac{1}{40}\left(\int_5^{25} 5 \, dt + \int_{25}^{35} 4 \, dt + \int_{35}^{45} 1 \, dt\right)$$

$$= \frac{1}{40}(100 + 40 + 10)$$

$$= \frac{15}{4}$$

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Question 5: [14%, Work-out question] Consider a DT periodic signal $x[n]$. We know that

$$x[n] = \sin(\frac{7\pi}{3}n) \tag{7}$$

Define a_k as the DTFS coefficients of $x[n]$.

1. [8%] Find the period N and the DTFS coefficients a_k of $x[n]$.

Consider another signal $y[n]$. We do not know the values of $y[n]$ explicitly. But we do know that $y[n]$ has period 6 and its DTFS coefficients are b_0 to b_5 being $b_k = k$. Namely, $b_0 = 0$, $b_1 = 1$, $b_2 = 2$, $b_3 = 3$, $b_4 = 4$, and $b_5 = 5$. Define $z[n] = x[n] \cdot y[n]$.

2. [6%] Find the DTFS coefficient c_2 of $z[n]$. Namely, you do not need to find the value for general c_k. Instead, we only need to find the c_k when $k = 2$.

   Hint 1: You would need the multiplication property of DTFS.

   Hint 2: If you do not know the answer to the previous question, you can assume $a_2 = a_4 = 3$, and $a_0 = a_1 = a_3 = a_5 = 0$. You will receive 5 points if your answer is correct.

Answer

5.1. $\frac{7\pi}{3}N = 2\pi m$, m is an arbitrary integer

the smallest possible N (fundamental period)

is $N=6$ (obtained when $m=7$)

$x[n] = \frac{1}{2j}e^{j\frac{7\pi}{3}n} - \frac{1}{2j}e^{-j\frac{7\pi}{3}n}$

$a_7 = \frac{1}{2j} = a_1$ $\quad a_{-7} = -\frac{1}{2j} = a_5$

The DTFS is

$a_1 = \frac{1}{2j}$, $a_5 = -\frac{1}{2j}$ $a_k = 0$ for $k = 0, 2, 3, 4$

periodic with $K = 6$.

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This sheet is for Question 5.

5.2. By the multiplication property

$$C_k = \sum_{l=\langle N\rangle} a_l b_{k-l}$$

$$C_2 = \langle a_0, a_1, a_2, a_3, a_4, a_5\rangle \cdot \langle b_2, b_1, b_0, b_{-1}, b_{-2}, b_{-3}\rangle$$

$$= \langle a_0, a_1, a_2, a_3, a_4, a_5\rangle \cdot \langle b_2, b_1, b_0, b_5, b_4, b_3\rangle$$

$$= \frac{1}{2j}\cdot 1 - \frac{1}{2j}\cdot 3$$

$$= j$$

Hint 2: $C_2 = 3 \cdot 0 + 3 \cdot 4 = 12.$

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Question 6: [20%, Multiple Choices]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \begin{cases} \int_0^{20} e^{-x_1(t-s)} ds & \text{if } x_1(t+1) > 0 \\ \int_0^{20} e^{+x_1(t-s)} ds & \text{if } x_1(t+1) \leq 0 \end{cases} \tag{8}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = \sum_{k=0}^{\infty} \cos(k-1) x_2[2n-k] \tag{9}$$

Answer the following questions

1. [4%] Is System 1 memoryless? Is System 2 memoryless?
2. [4%] Is System 1 causal? Is System 2 causal?
3. [4%] Is System 1 stable? Is System 2 stable?
4. [4%] Is System 1 linear? Is System 2 linear?
5. [4%] Is System 1 time-invariant? Is System 2 time-invariant?

Answer:

Sys 1:	Sys 2:
not memoryless	not memoryless
not causal	not causal
stable	not stable
not linear	linear
time invariant	not time invariant