PURDUE

NAME _____

PUID _____

Tips for making sure GradeScope can read your exam:

1. Make sure your name and PUID are clearly written at the top of every page, including any additional blank pages you use.
2. Write only on the front of the exam pages.
3. Add any additional pages used to the back of the exam before turning it in.
4. Ensure that all pages are facing the same direction.
5. Answer all questions in the area designated for that answer. Do not run over into the next question space.

1. <u>**Do not write answers on the back of pages!**</u>

2. **After the exam ended, you will have 5 additional minutes to write down your name and Purdue ID on each of the pages.**

3. **If you need additional sheets to write down your answers, please let the instructor/TA know. We will hand out additional answer sheets then.**

4. Enter your student ID number, and signature in the space provided on this page.

5. This is a closed book exam.

6. This exam contains multiple-choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have two hours to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.

7. The instructor/TA will hand out loose sheets of paper for the rough work.

8. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [20%, Work-out question]

1. [1%] What does the acronym AM-SSB stand for?

Prof. Wang wanted to transmit an AM-SSB signal. To that end, he wrote the following MATLAB code.

```
% Initialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read two different .wav files
[x1, f_sample, N]=audioread('x1.wav');
x1=x1';
[x2, f_sample, N]=audioread('x2.wav');
x2=x2';

% Step 0: Initialize several parameters
W_1=????;
W_2=4500*pi;
W_3=????;
W_4=????;
W_5=????;
W_6=2000*pi;
W_7=7000*pi;


% Step 1: Make the signals band-limited.
h=1/(pi*t).*(sin(W_1*t));
x1_new=ece301conv(x1, h);
x2_new=ece301conv(x2, h);

% Step 2: Multiply x1_new and x2_new with a sinusoidal wave.
x1_h=x1_new.*cos(W_2*t);
x2_h=x2_new.*sin(W_3*t);

% Step 3: Keep one of the two side bands
h_one=1/(pi*t).*(sin(W_4*t)-sin(W_5*t));
h_two=1/(pi*t).*(sin(W_6*t).*(2*cos(W_7*t)));
x1_sb=ece301conv(x1_h, h_one);
```

```
x2_sb=ece301conv(x2_h, h_two);

% Step 4: Create the transmitted signal
y=x1_sb+x2_sb;
audiowrite('y.wav', y, f_sample);
```

2. [1%] What is the carrier frequency (Hz) of the signal x1_new?

3. [2%] Our goal is to transmit the "lower-side bands" of the x2 signal. What should be the value of W_3 in the MATLAB code?

4. [2%] At the same time, I would also like to "maximize the quality" of the x2 signal. That is, I would like to choose the largest possible W_1 value so that I can keep as much initial frequency as possible. What is the largest W_1 value that is consistent with the choices of W_6 and W_7 values?

5. [4%] Our goal is to also transmit the "lower-side bands" for the x1 signal. What should be the values of W_4 and W_5 in the MATLAB code, assuming the same W_1 used in the previous sub-question?

6. [1.5%] Suppose I would like to "pack the two radio stations as closely together as possible", but I must not change the W_6 and W_7 given in the above code and also must not change the W_1 value given in Q1.4. What is the largest W_2 value I can choose without worry about any issue of frequency overlap?

Prof. Wang decided to use the lower-side-band transmission with W_2 set to 4500π and used the code in the previous page to generate the "y.wav" file. (That is, we do not pack the two radio stations as closely as possible and simply ignore Q1.7.)

A student tried to demodulate the output waveform "y.wav" by the following code.

```
% Initialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read the .wav files
[y, f_sample, N]=audioread('y.wav');
y=y';

% Initialize several parameters


W_8=????;
W_9=????;
W_10=????;



% We reuse the W4 to W7 values in the transmitter design

W_4=????;
W_5=????;
W_6=2000*pi;
W_7=7000*pi;


% Create the low-pass filter.
h_M=1/(pi*t).*(sin(W_8*t));

% We reuse
h_one=1/(pi*t).*(sin(W_4*t)-sin(W_5*t));
h_two=1/(pi*t).*(sin(W_6*t).*(2*cos(W_7*t)));

% demodulate signal 1
y11=ece301conv(y, h_one);
y1=y11.*sin(W_9*t);
x1_hat=4*ece301conv(y1,h_M);
```

```
sound(x1_hat,f_sample)

% demodulate signal 2
y21=ece301conv(y, h_two);
y2=y21.*sin(W_10*t);
x2_hat=4*ece301conv(y2,h_M);

sound(x2_hat,f_sample)
```

7. [4.5%] Continue from the previous questions. What should the values of W_8 to W_10 be in the MATLAB code?

8. [4%] It turns out that the above MATLAB code is not written correctly and the end results do not sound right. Answer the following questions

   (a) Is signal x1_new correctly/perfectly demodulated? If yes, then go to subquestion (d). If no, then continue answering the following sub-questions.

   (b) Use 2 to 3 sentences to answer (i) what kind of problem does x1_new have, i.e., how does the problem impact the sound quality of "sound(x1_hat,f_sample)"?

   (c) How can the MATLAB code be corrected so that the playback/demodulation can be successful?

   (d) Is signal x2_new correctly/perfectly demodulated? If yes, then your answer to Q1.8 is complete. If no, then continue answering the following sub-questions.

   (e) Use 2 to 3 sentences to answer (i) what kind of problem does x2_new have, i.e., how does the problem impact the sound quality of "sound(x2_hat,f_sample)"?

   (f) How can the MATLAB code be corrected so that the playback/demodulation can be successful?

   Hint: If you do not know the answers of Q1.2 to Q1.8, please simply draw the AMSSB modulation (using lower side band) and demodulation diagrams and mark carefully all the parameter values. You will receive 10 points for Q1.2 to Q1.8 if your system diagrams are correct and all parameter values are marked correctly.

Last Name: First Name: Purdue ID:

This sheet is for Question 1.

Last Name: First Name: Purdue ID:
This sheet is for Question 1.

Question 2: [8%, Work-out question]

Consider a continuous time AM signal $y(t) = x(t)\cos(400\pi t)$. Namely the carrier frequency is $\omega_0 = 400\pi$. We also know that the frequency domain of the AM signal $y(t)$ is

$$Y(j\omega) = \begin{cases} 0.5 & \text{if } 399\pi \leq |\omega| \leq 401\pi \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

1. [2.5%] Suppose we use the *synchronous demodulation* to reconstruct the original signal $x(t)$ from $y(t)$ and we denote the output by $x_{\text{syn}}(t)$. Plot $x_{\text{syn}}(t)$ for the range of $-3 \leq t \leq 3$.

2. [5.5%] Suppose we use the *asynchronous demodulation* to reconstruct the original signal $x(t)$ from $y(t)$ and we denote the output by $x_{\text{asynch}}(t)$. Plot $x_{\text{asynch}}(t)$ for the range of $-3 \leq t \leq 3$.

   Hint 1: You may want to draw $y(t)$ first, which may help your answer this sub-question.

   Hint 2: If you do not know the answer to the previous sub-question, please answer the following instead (i) Plot $\frac{\sin(2t)}{4t}$ for the range of $-5\pi \leq t \leq 5\pi$. Please carefully marked all the critical points; and (ii) What is the main difference between *synchronous* demodulation and *asynchronous* demodulation. You will receive 3 points if both your answers of (i) and (ii) are correct.

Last Name: First Name: Purdue ID:
This sheet is for Question 2.

Last Name: First Name: Purdue ID:

This sheet is for Question 2.

Question 3: [14%, Work-out question]

1. [4%] Consider a continuous time signal

$$x(t) = \begin{cases} \cos(5\pi t) & \text{if } 0.1 \le |t| \le 0.3 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

   We sample $x(t)$ with the sampling frequency 10Hz and denote the sampled values by $x[n]$. Plot $x[n]$ for the range of $-5 \le n \le 5$.

2. [5%] Find the expression of the DTFT $X(e^{j\omega})$ of $x[n]$. Plot $X(e^{j\omega})$ for the range of $-2\pi < \omega < 2\pi$.

   Hint: If you do not know the answer of $x[n]$, you can assume that $x[n] = \sum_{k=2}^{3} \delta[n - k] + \delta[n + k]$. You will receive full points if your answer is correct.

3. [3%] We use $x_{\text{ZOH}}(t)$ to represent the reconstructed signal using "zero-order hold". Plot $x_{\text{ZOH}}(t)$ for the range of $-1 \le t \le 1$.

   Hint: If you do not know the answer of $x[n]$, you can assume that $x[n] = \sum_{k=2}^{3} \delta[n - k] + \delta[n + k]$. You will receive full points if your answer is correct.

4. [2%] A student tried to duplicate the results of the previous sub-question using the same sampled array $x[n]$. However, when he/she tried to reconstruct the signal, he/she mistakenly set the sampling frequency to be 5Hz. We use $\hat{x}_{\text{ZOH}}(t)$ to represent the latest reconstructed signal using by him/her. That is, both the previous sub-question $x_{\text{ZOH}}(t)$ and this sub-question $\hat{x}_{\text{ZOH}}(t)$ apply ZOH to the same sampled array $x[n]$. However, the former uses the correct sampling frequency 10Hz but the latter uses a wrong sampling frequency 5Hz. Plot $\hat{x}_{\text{ZOH}}(t)$ for the range of $-1 \le t \le 1$.

   Hint: If you do not know the answer of $x[n]$, you can assume that $x[n] = \sum_{k=2}^{3} \delta[n - k] + \delta[n + k]$. You will receive full points if your answer is correct.

Last Name: First Name: Purdue ID:

This sheet is for Question 3.

This sheet is for Question 3.

Question 4: [16%, Work-out question]

1. [1%] Consider a continuous time signal $x(t) = 2 + \cos(\frac{5\pi t}{3})$. Plot $x(t)$ for the range of $-1.2 < t < 1.2$.

2. [5%] We sample $x(t)$ via *impulse train sampling* with sampling period 0.3. Denote the final *impulse-train-sampled* signal by $x_p(t)$. Plot $x_p(t)$ for the range of $-1.2 < t < 1.2$;

3. [5%] Consider an LTI system S1 with impulse response

$$h_1(t) = \begin{cases} \frac{t}{0.3} + 1 & \text{if } -0.3 < t \le 0 \\ 1 - \frac{t}{0.3} & \text{if } 0 < t \le 0.3 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

   Let $y_1(t)$ denote the output of this system when using $x_p(t)$ as the input. Plot $y_1(t)$ for the range of $-1.2 < t < 1.2$.

4. [5%] Plot the Fourier transform $X_p(j\omega)$ for the range of $-\frac{20\pi}{3} < \omega < \frac{20\pi}{3}$.

   Hint: You can still answer Q4.4 even if you do not know the answer to Q4.3.

Last Name: First Name: Purdue ID:

This sheet is for Question 4.

Last Name: First Name: Purdue ID:

This sheet is for Question 4.

Question 5: [10%, Work-out question]

Consider the following discrete time signals

$$x[n] = \cos(\frac{13\pi n}{5}) \tag{4}$$

$$y[n] = e^{\frac{j3\pi n}{5}} \tag{5}$$

Define $z[n] = x[n] \cdot y[n]$.

1. [4%] Find the DTFS a_k of $x[n]$.

2. [1.5%] Find the DTFS b_k of $y[n]$.

3. [4.5%] Find the DTFS c_k of $z[n]$.

   Hint: If you do not know the a_k and b_k of the previous two sub-questions, you can express c_k as an expression of a_k and b_k. You will receive 2 points if your answer is correct.

Last Name: First Name: Purdue ID:

This sheet is for Question 5.

Question 6: [8%, Work-out question]

Consider the following continuous time input and continuous time output system.

$$y(t) = \int_{t-2}^{t+1} (s - t - 1)x(s - 3)ds \tag{6}$$

Find the impulse response $h(t)$ of this system and plot it for the range of $-6 < t < 6$.

Last Name: First Name: Purdue ID:

This sheet is for Question 6.

Last Name: First Name: Purdue ID:

This sheet is for Question 6.

Question 7: [9%, Work-out question]

Consider a CT signal $x(t)$ with the corresponding Fourier transform being

$$X(j\omega) = \frac{4}{-3 + j\omega} \tag{7}$$

The following sub-questions aim to find the $x(t)$.

1. [3%] Consider a new signal $y(t) = x(-t)$. Find the CTFT $Y(j\omega)$;

2. [2%] Find the expression of $y(t)$.

3. [3%] Find the expression of $x(t)$ and plot it for the range of $-3 < t < 3$.

4. [1%] We know from Table 4.2 Basic Fourier Transform Pairs:

   Time domain: $e^{-at}u(t), \mathrm{Re}(a) > 0$; Frequency domain: $\frac{1}{a+j\omega}$.

   If we apply this formula blindly, we will have $x(t) = 4e^{-(-3)t}u(t)$ by comparing Equation (7) with the above Fourier transform pair.

   Answer this question: Why this direct application of the table <u>does not</u> give the right answer of $x(t)$. Please write a few sentence about why blindly applying the formula does not give us the right answer. Specifically, your answer should be about why we require $\mathrm{Re}(a) > 0$ in the formula table above.

   Hint: you should think about how we find the Fourier transform of $e^{-at}u(t)$.

This sheet is for Question 7.

This sheet is for Question 7.

Question 8: [15%, Multiple-choice question] Consider two signals

$$h_1(t) = \int_0^{3+\sin(t)} s^2 \cos s \, ds \tag{8}$$

and

$$h_2[n] = \sum_{k=-\infty}^{\infty} U[(k-3)^3] \cdot e^{-(k-3)\cos^2(\pi(k-3))} \cdot \delta[n-(k-3)] \tag{9}$$

1. [1.25%] Is $h_1(t)$ periodic?

2. [1.25%] Is $h_2[n]$ periodic?

3. [1.25%] Is $h_1(t)$ even or odd or neither?

4. [1.25%] Is $h_2[n]$ even or odd or neither?

5. [1.25%] Is $h_1(t)$ of finite power?

6. [1.25%] Is $h_2[n]$ of finite power?

Suppose the above two signals are also the impulse responses of two LTI systems: System 1 and System 2, respectively.

1. [1.25%] Is System 1 memoryless?

2. [1.25%] Is System 2 memoryless?

3. [1.25%] Is System 1 causal?

4. [1.25%] Is System 2 causal?

5. [1.25%] Is System 1 stable?

6. [1.25%] Is System 2 stable?

Last Name: First Name: Purdue ID:

This sheet is for Question 8.

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

TABLE 3.1   PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$ $\Big\}$ Periodic with period T and $y(t)$ $\Big\}$ fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t),\ \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\displaystyle\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\displaystyle\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\dfrac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\dfrac{2\pi}{T}a_k$
Integration		$\displaystyle\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\dfrac{1}{jk\omega_0}\right)a_k = \left(\dfrac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ |a_k| = |a_{-k}| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t)\text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t)\text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t - 1) - 1/2.$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2   PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$x[n]$ ⎱ Periodic with period N and $y[n]$ ⎰ fundamental frequency $\omega_0 = 2\pi/N$	a_k ⎱ Periodic with b_k ⎰ period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic}\\\text{with period } mN\end{pmatrix}$
Periodic Convolution	$\displaystyle\sum_{r=\langle N\rangle} x[r]y[n - r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\displaystyle\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n - 1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\displaystyle\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only}\\\text{if } a_0 = 0\end{pmatrix}$	$\left(\dfrac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ |a_k| = |a_{-k}| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & \text{[x[n] real]} \\ x_o[n] = \mathcal{Od}\{x[n]\} & \text{[x[n] real]} \end{cases}$	$\mathfrak{Re}\{a_k\}$ $j\mathfrak{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2 = \sum_{k=\langle N\rangle}|a_k|^2$$

4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(j\omega)$
		$y(t)$	$Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\dfrac{1}{|a|}X\left(\dfrac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\dfrac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\dfrac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\dfrac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\dfrac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ |X(j\omega)| = |X(-j\omega)| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

4.3.7	Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2 d\omega$$

, we have consid-
re summarized in
which each prop-

important Fourier
pply the tools of

transform

$i\omega)$

$) - \theta))d\theta$

$'(0)\delta(\omega)$

$-j\omega)$
$\mathcal{R}e\{X(-j\omega)\}$
$-\mathcal{I}m\{X(-j\omega)\}$
$-j\omega)|$
$\angle X(-j\omega)$
ven

iginary and odd

TABLE 4.2   BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\displaystyle\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$\displaystyle 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0,\quad$ otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0,\quad$ otherwise
$\sin\omega_0 t$	$\dfrac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0,\quad$ otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1,\quad a_k = 0,\ k \neq 0$ $\left(\begin{array}{l}\text{this is the Fourier series representation for}\\\text{any choice of } T > 0\end{array}\right)$
Periodic square wave $x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \le \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\displaystyle\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\,\delta(\omega - k\omega_0)$	$\dfrac{\omega_0 T_1}{\pi}\,\mathrm{sinc}\!\left(\dfrac{k\omega_0 T_1}{\pi}\right) = \dfrac{\sin k\omega_0 T_1}{k\pi}$
$\displaystyle\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\dfrac{2\pi}{T} \displaystyle\sum_{k=-\infty}^{+\infty} \delta\!\left(\omega - \dfrac{2\pi k}{T}\right)$	$a_k = \dfrac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \|t\| < T_1 \\ 0, & \|t\| > T_1 \end{cases}$	$\dfrac{2\sin\omega T_1}{\omega}$	—
$\dfrac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \|\omega\| < W \\ 0, & \|\omega\| > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\dfrac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t),\ \mathcal{R}e\{a\} > 0$	$\dfrac{1}{a + j\omega}$	—
$te^{-at}u(t),\ \mathcal{R}e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^2}$	—
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t),$ $\mathcal{R}e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^n}$	—

TABLE 5.1   PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$	$X(e^{j\omega})$ ⎤ periodic with
		$y[n]$	$Y(e^{j\omega})$ ⎦ period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\displaystyle\sum_{k=-\infty}^{n} x[k]$	$\dfrac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+ \pi X(e^{j0})\displaystyle\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ |X(e^{j\omega})| = |X(e^{-j\omega})| \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals $\displaystyle\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \dfrac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2 d\omega$		

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

5.7.1 Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients a_k of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence a_k in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence a_k are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\displaystyle\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$\displaystyle 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$\displaystyle 2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $\quad a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos\omega_0 n$	$\displaystyle \pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $\quad a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin\omega_0 n$	$\displaystyle \frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $\quad a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$\displaystyle 2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & N_1 < |n| \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$\displaystyle 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \dfrac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \ne 0, \pm N, \pm 2N, \dots$ $a_k = \dfrac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \dots$
$\displaystyle\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\displaystyle \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \dfrac{1}{N}$ for all k
$a^n u[n], \quad |a| < 1$	$\dfrac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}$	$\dfrac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\dfrac{\sin Wn}{\pi n} = \dfrac{W}{\pi} \operatorname{sinc}\left(\dfrac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le |\omega| \le W \\ 0, & W < |\omega| \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\dfrac{1}{1 - e^{-j\omega}} + \displaystyle\sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \quad |a| < 1$	$\dfrac{1}{(1 - ae^{-j\omega})^2}$	—
$\dfrac{(n + r - 1)!}{n!(r - 1)!} a^n u[n], \quad |a| < 1$	$\dfrac{1}{(1 - ae^{-j\omega})^r}$	—

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\dfrac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $|a|R$ = the set of points $\{|a|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $|z| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\dfrac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $|z| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R
10.5.9	Initial Value Theorem			

Initial Value Theorem
If $x[n] = 0$ for $n < 0$, then
$$x[0] = \lim_{z \to \infty} X(z)$$

TABLE 10.2   SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$|z|>1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$|z|<1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m>0$) or ∞ (if $m<0$)
5. $\alpha^n u[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$|z|>|\alpha|$
6. $-\alpha^n u[-n-1]$	$\dfrac{1}{1-\alpha z^{-1}}$	$|z|<|\alpha|$
7. $n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$|z|>|\alpha|$
8. $-n\alpha^n u[-n-1]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$|z|<|\alpha|$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$|z|>1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$|z|>1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$|z|>r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$|z|>r$

10.7.1 Causality

A causal LTI system has an impulse response $h[n]$ that is zero for $n<0$, and therefore is right-sided. From Property 4 in Section 10.2 we then know that the ROC of $H(z)$ is the exterior of a circle in the z-plane. For some systems, e.g., if $h[n]=\delta[n]$, so that $H(z)=1$, the ROC can extend all the way in to and possibly include the origin. Also, in general, for a right-sided impulse response, the ROC may or may not include infinity. For example, if $h[n]=\delta[n+1]$, then $H(z)=z$, which has a pole at infinity. However, as we saw in Property 8 in Section 10.2, for a causal system the power series

$$H(z)=\sum_{n=0}^{\infty}h[n]z^{-n}$$

does not include any positive powers of z. Consequently, the ROC includes infinity. Summarizing, we have the follow principle:

A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity.