2020 MT3 Solution

Q1) 1. $j^k = e^{j\frac{\pi}{2}k} = e^{j\frac{2\pi}{40}k\cdot 10}$ Period $N=40$ for $x[n]$

$$\sum_{k=0}^{39} a_k j^k = \sum_{k=0}^{39} a_k \cdot e^{j\frac{2\pi}{40}k\cdot 10}$$

*Recall $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}\cdot n}$

$$\Longrightarrow \sum_{k=0}^{39} a_k \cdot e^{j\frac{2\pi}{40}k\cdot 10} = x[10] = \boxed{2-3^{-10}}$$

2. By Parseval's Relation : $\sum_{k=0}^{39} |a_k|^2 = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2$

$$= \frac{1}{40}\sum_{n=1}^{40} |x[n]|^2$$

$$= \frac{1}{40}\sum_{n=1}^{20} |2-3^{-n}|^2 + \frac{1}{40}\sum_{n=21}^{40} |2|^2$$

$$= \frac{1}{40}\sum_{n=1}^{20} (2-3^{-n})^2 + \frac{1}{40}\cdot 20\cdot 4$$

$$= \frac{1}{40}\sum_{n=1}^{20} (4-4\cdot 3^{-n}+3^{-2n}) + 2$$

$$= \frac{1}{40}\cdot 20\cdot 4 - \frac{1}{40}\cdot 4\cdot \sum_{n=1}^{20} 3^{-n} + \frac{1}{40}\sum_{n=1}^{20}\left(\frac{1}{9}\right)^n + 2$$

$$= 2 - \frac{1}{10}\sum_{n=1}^{20}\left(\frac{1}{3}\right)^{n-1+1} + \frac{1}{40}\sum_{n=1}^{20}\left(\frac{1}{9}\right)^{n-1+1} + 2$$

$$= 4 - \frac{1}{10}\cdot\frac{1}{3}\cdot\sum_{n=1}^{20}\left(\frac{1}{3}\right)^{n-1} + \frac{1}{40}\cdot\frac{1}{9}\cdot\sum_{n=1}^{20}\left(\frac{1}{9}\right)^{n-1}$$

$$= 4 - \frac{1}{30}\cdot\frac{1-\left(\frac{1}{3}\right)^{20}}{1-1/3} + \frac{1}{360}\cdot\frac{1-\left(\frac{1}{9}\right)^{20}}{1-1/9}$$

$$= \boxed{4 - \frac{1-\frac{1}{3^{20}}}{20} + \frac{1-\frac{1}{9^{20}}}{320}}$$

Q2) 1. For $\cos(3k\pi t)$, Period $T_0 = \frac{2\pi}{3k\pi} = \frac{2}{3k}$

Considering $k = 1, 2, \ldots, 15$, find LCM of all signals' periods:

$$T = \text{LCM}\left\{\frac{2}{3}, \frac{2}{6}, \frac{2}{9}, \ldots, \frac{2}{45}\right\} = \frac{2}{3}$$

$$\Rightarrow \boxed{T = \frac{2}{3}}$$

2. To avoid confusion, let's rewrite $x(t) = \sum_{s=1}^{15} \cos(3s\pi t)$

For a given value of s, the FS coefficients of $\cos(3s\pi t)$ are given by:

$$\cos(3s\pi t) = \frac{1}{2}e^{j3s\pi t} + \frac{1}{2}e^{-j3s\pi t}$$

$$= \frac{1}{2}e^{js\frac{2\pi}{T}t} + \frac{1}{2}e^{-js\frac{2\pi}{T}t} \quad \text{where } T = \frac{2}{3}$$

$$\Rightarrow a_k = \tfrac{1}{2} \text{ if } k = \pm s, \ 0 \text{ if } k \neq \pm s$$

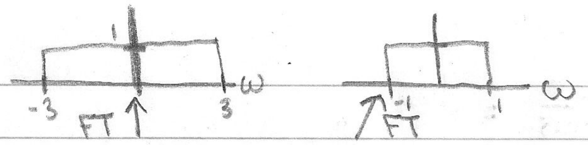
$$= \tfrac{1}{2}\delta(k-s) + \tfrac{1}{2}\delta(k+s)$$

So for full signal $x(t)$: $\boxed{a_k = \sum_{s=1}^{15}\left(\tfrac{1}{2}\delta(k-s) + \tfrac{1}{2}\delta(k+s)\right)}$

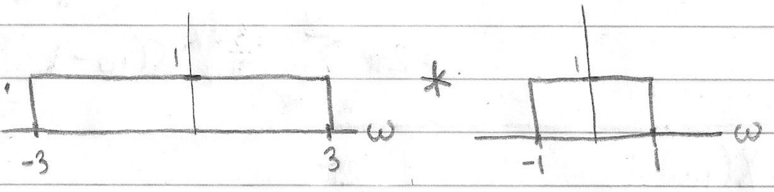
3. $h(t) = e^{-2t}u(t) \longleftrightarrow H(j\omega) = \frac{1}{2+j\omega}$

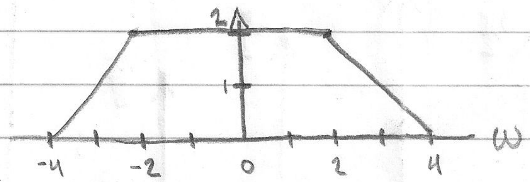
$$H(jk\omega_0) = H(jk3\pi) = \frac{1}{2+jk3\pi}$$

$$b_k = a_k H(j\omega_0 k) = \boxed{\frac{1}{2}\sum_{s=1}^{15} \frac{\delta(k-s)}{2+jk3\pi} + \frac{\delta(k+s)}{2+jk3\pi}}$$

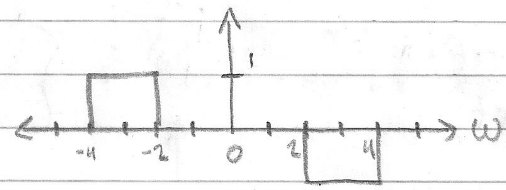


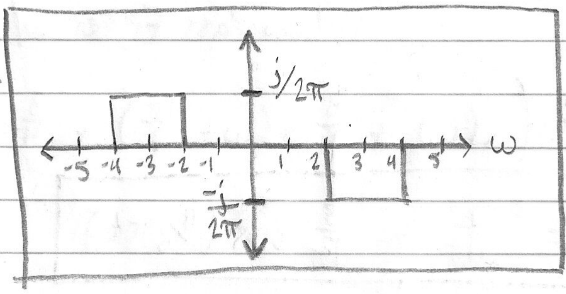
Q3) $x(t) = \frac{\sin(3t)}{\pi t} \cdot \frac{\sin(t)}{\pi t}$

$X(j\omega) = \frac{1}{2\pi} \cdot$ 

$= \frac{1}{2\pi} \cdot$ 

$y(t) = t \cdot x(t) \Rightarrow Y(j\omega) = j \, {}^{d}/_{d\omega} X(j\omega)$

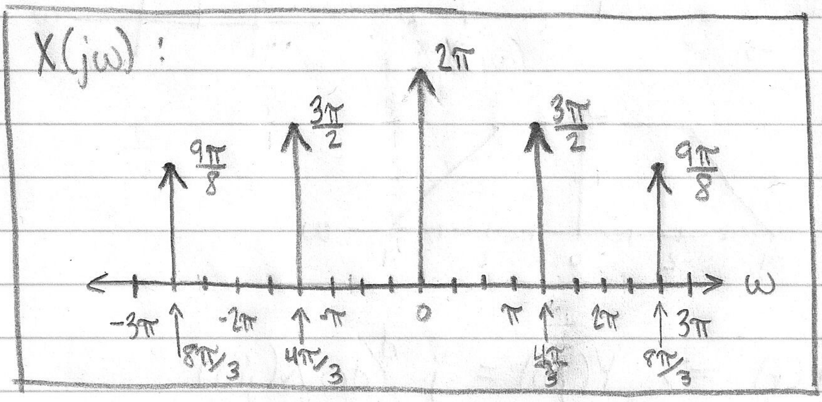
$\Rightarrow Y(j\omega) = \frac{j}{2\pi} \cdot$ 

$\Rightarrow Y(j\omega) =$ 

Q4) 1. $T = 1.5 \Rightarrow \omega_0 = \frac{2\pi}{1.5} = \frac{4\pi}{3}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \xleftrightarrow{FT} H(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

$$= 2\pi \sum_{k=-\infty}^{\infty} \left(\frac{3}{4}\right)^{|k|} \delta\left(\omega - k \cdot \frac{4\pi}{3}\right)$$

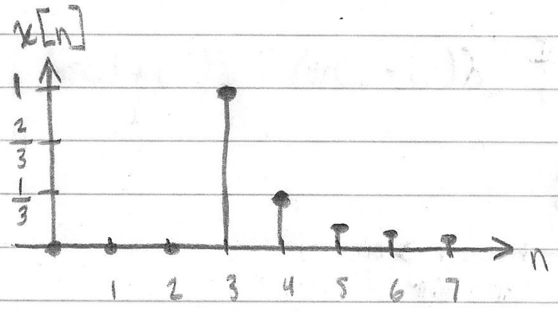


2. $h(t) = \frac{\sin(2\pi t)}{\pi t} \xleftrightarrow{FT} H(j\omega) = \begin{cases} 1 & |\omega| < 2\pi \\ 0 & |\omega| > 2\pi \end{cases}$

Low Pass Filter w/ cutoff $\omega = 2\pi$

$$\Rightarrow Y(j\omega) = 2\pi\left(\delta(\omega) + \frac{3}{4}\delta\left(\omega + \frac{4\pi}{3}\right) + \frac{3}{4}\delta\left(\omega - \frac{4\pi}{3}\right)\right)$$

$$\Rightarrow \boxed{y(t) = 1 + \frac{3}{2}\cos\left(\frac{4}{3}\pi t\right)}$$

Q5) 

$= \begin{cases} 3^{-n} & \text{if } n \geq 0 \\ 0 & \text{else} \end{cases}$ shifted right by 3

let this equal $x_1[n]$

$$X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{\infty} 3^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{3} e^{-j\omega}\right)^n$$

$$= \frac{1}{1 - \frac{1}{3} e^{-j\omega}}$$

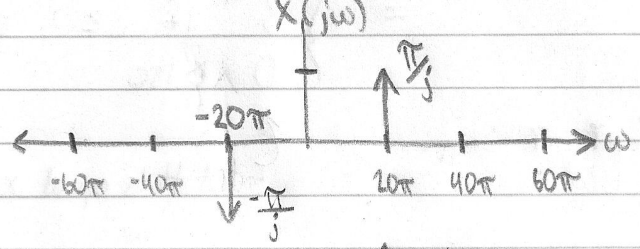
By time shifting property of DTFT:

$$X(j\omega) = e^{-j\omega 3} \cdot X_1(j\omega) = \frac{e^{-3j\omega}}{1 - \frac{1}{3} e^{-j\omega}}$$

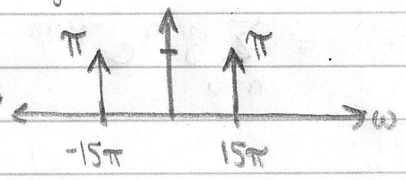
$$\Rightarrow X(j0.5\pi) = \frac{e^{-3j \cdot 0.5\pi}}{1 - \frac{1}{3} e^{-j \cdot 0.5\pi}} = \frac{e^{-3/2 j\pi}}{1 - \frac{1}{3} e^{-j\pi/2}}$$

$$= \frac{j}{1 + \frac{1}{3} j} \cdot \frac{j}{j} = \frac{-1}{j - \frac{1}{3}} = \boxed{\frac{1}{\frac{1}{3} - j}}$$

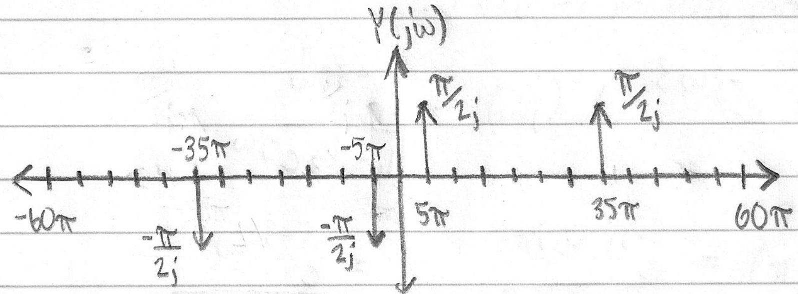
Q6) 1. $x(t) = \sin(20\pi t) \xrightarrow{FT} \frac{\pi}{j}\left[\delta(\omega - 20\pi) + \delta(\omega + 20\pi)\right] = X(j\omega)$



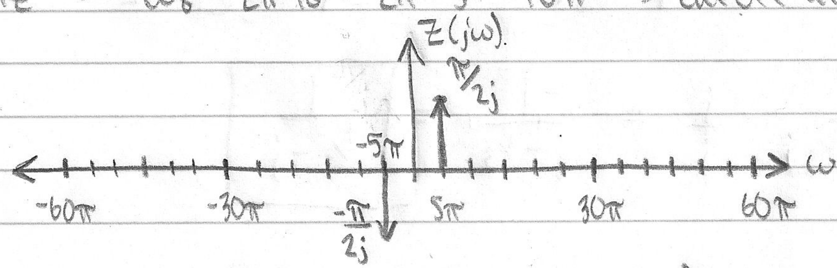
2. $\cos(15\pi t) \xrightarrow{FT}$



$Y(j\omega) = FT\{x(t)\} * FT\{\cos(15\pi t)\} \cdot \frac{1}{2\pi}$



3. $5\,Hz \rightarrow \omega_0 = 2\pi f_0 = 2\pi \cdot 5 = 10\pi \rightarrow$ cutoff at $10\pi = \omega$



4. $Z(j\omega) = \frac{1}{2} \cdot \frac{\pi}{j}\left(\delta(\omega - 5\pi) - \delta(\omega + 5\pi)\right)$

$\downarrow$ IFT

$$\boxed{z(t) = \tfrac{1}{2}\sin(5\pi t)}$$