MT2 Fall 2020

Question 1 Consider the following DT-Linear System

$$y[n] = \sum_{k=-|n|}^{2-|n|} x[k+2]\cdot|k|$$

1.1 To find the impulse response of an LTI system, Let $x[n] = \delta[n]$ as input

$$h[n] = \sum_{k=-|n|}^{2-|n|} \delta[k+2]\cdot|k| \qquad \text{Since} \quad \delta[k+2] = \begin{cases} 1 & k=-2 \\ 0 & \text{otherwise} \end{cases}, \quad \text{we have}$$

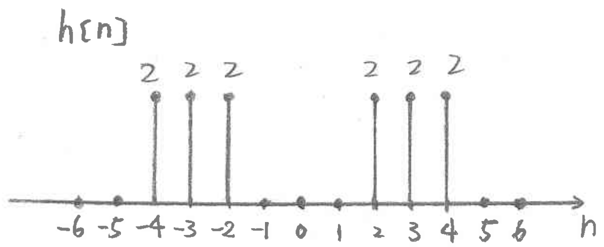
The summation window $[-|n|, 2-|n|]$ has length 3 : $-|n|$, $1-|n|$, $2-|n|$

We verify if the window $k = -|n|, 1-|n|, 2-|n|$ cover $k=-2$

Case 1: $2-|n| < -2 \Rightarrow |n| > 4$, the summation window does not cover $k=-2 \Rightarrow h[n]=0$

Case 2: $-2 \le 2-|n| \le 0 \Rightarrow 2 \le |n| \le 4$, the summation window covers $k=-2 \Rightarrow h[n] = \delta[0]\cdot|-2| = 2$

Case 3: $-|n| > -2 \Rightarrow |n| < 2$, the summation window does not cover $k=-2 \Rightarrow h[n]=0$



1.2 The system is time variant. Let $x_1[n] = \delta[n-1]$, the output of system

$$y_1[n] = \sum_{k=-|n|}^{2-|n|} \delta[k+1]\cdot|k| = \begin{cases} 1 & -3 \le n \le -1,\ 1 \le n \le 3 \\ 0 & \text{otherwise} \end{cases} \neq h[n-1]$$

Therefore for $x[n] = \delta[n] + \delta[n-1]$, system output $y[n] \neq x[n] * h[n]$

$\Rightarrow$ False, the equality does not hold.

1.3 (i) A system is invertible if given the output $y(t)$, the input $x(t)$ can be deduced

(ii) To prove a system is not invertible, we need to show there exists two different inputs $x_1[n] \neq x_2[n]$ leading to same outputs $y_1[n] = y_2[n]$

(iii) Consider the input $x_1[n] = \delta[n-2]$, the system output is

$$y_1[n] = \sum_{k=-|n|}^{2-|n|} \delta[k]\cdot|k| \qquad \text{where } \delta[k] = \begin{cases} 1 & k=0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore $\delta[k]\cdot|k| = 0$ for all k $\Rightarrow$ $y_1[n] = 0$ for all n

Consider another input $x_2[n] = 0$, the system output $y_2[n] = 0$

Since $x_1 = \delta[n-2] \neq 0 = x_2[n]$ and $y_1[n] = y_2[n] = 0$, we show that the system is NOT invertible

Question 2 Consider an LTI system with impulse response being $h(t) = \delta(t-4\sqrt{2}) + e^{-t}U(t+1)$

Let input $x(t) = \cos(3t+1.5)$ and system be LTI, the output

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(s)\,x(t-s)\,ds$$

$$= \int_{-\infty}^{\infty} \left[\delta(s-4\sqrt{2}) + e^{-s}U(s+1)\right] \cos(3t-3s+\tfrac{3}{2})\,ds$$

$$= \int_{-\infty}^{\infty} \delta(s-4\sqrt{2}) \cos(3t-3s+\tfrac{3}{2})\,ds + \int_{-\infty}^{\infty} e^{-s}U(s+1)\cos(3t-3s+\tfrac{3}{2})\,ds$$

$$= \int_{-\infty}^{\infty} \cos(3t-12\sqrt{2}+\tfrac{3}{2})\,\delta(s-4\sqrt{2})\,ds + \int_{-1}^{\infty} e^{-s}\,\frac{e^{j(3t-3s+\frac{3}{2})} + e^{-j(3t-3s+\frac{3}{2})}}{2}\,ds$$

$$= \cos(3t-12\sqrt{2}+\tfrac{3}{2}) \int_{-\infty}^{\infty} \delta(s-4\sqrt{2})\,ds + \frac{1}{2}\int_{-1}^{\infty} e^{j(3t+\frac{3}{2})} e^{(-1-3j)s} + e^{-j(3t+\frac{3}{2})} e^{(-1+3j)s}\,ds$$

$$= \cos(3t-12\sqrt{2}+\tfrac{3}{2}) + \frac{1}{2} e^{j(3t+\frac{3}{2})} \cdot \frac{e^{1+3j}}{1+3j} + \frac{1}{2} e^{-j(3t+\frac{3}{2})} \frac{e^{1-3j}}{1-3j}$$

Alternative Use CTFT for $x(t) = \cos(3t+1.5)$ and $h(t) = \delta(t-4\sqrt{2}) + e^{-t}U(t+1)$

$$x(t) = \frac{1}{2}\left(e^{j(3t+1.5)} + e^{-j(3t+1.5)}\right) = \frac{e^{j3/2}}{2} e^{j3t} + \frac{e^{-j3/2}}{2} e^{j(-3)t}$$

$$X(j\omega) = \frac{e^{j3/2}}{2}\delta(\omega-3) + \frac{e^{-j3/2}}{2}\delta(\omega+3)$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\,e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}\left(\delta(t-4\sqrt{2}) + e^{-t}U(t+1)\right)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-j\omega 4\sqrt{2}}\,\delta(t-4\sqrt{2})\,dt + \int_{-1}^{\infty} e^{-(1+j\omega)t}\,dt$$

$$= e^{-j\omega 4\sqrt{2}} \int_{-\infty}^{\infty} \delta(t-4\sqrt{2})\,dt + \left.\frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\right|_{-1}^{\infty}$$

$$= e^{-j\omega 4\sqrt{2}} + \frac{e^{1+j\omega}}{1+j\omega}$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{e^{j3/2}}{2}\left(e^{-j12\sqrt{2}} + \frac{e^{1+3j}}{1+3j}\right)\delta(\omega-3) + \frac{e^{-j3/2}}{2}\left(e^{j12\sqrt{2}} + \frac{e^{1-3j}}{1-3j}\right)\delta(\omega+3)$$

$$y(t) = \frac{1}{2}\left(e^{-j12\sqrt{2}} + \frac{e^{1+3j}}{1+3j}\right)e^{j(\frac{3}{2}+3t)} + \frac{1}{2}\left(e^{j12\sqrt{2}} + \frac{e^{1-3j}}{1-3j}\right)e^{-j(\frac{3}{2}+3t)}$$

Question 3 Consider a DT-LTI system, when input $x[n] = u[n]$, the output is

$$y[n] = \begin{cases} n+1 & 0 \le n \le 3 \\ 7-n & 4 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$$

The impulse response is the system output with input $\delta[n]$

We can express $\delta[n] = u[n] - u[n-1]$

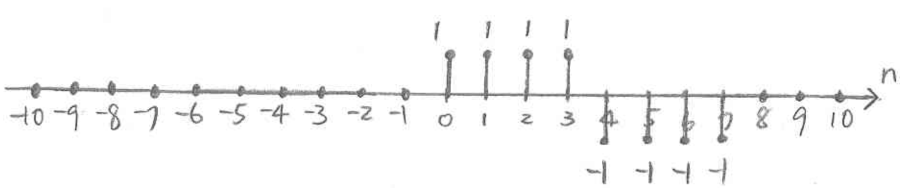
Since the system is LTI, we then have impulse response

$h[n] = y[n] - y[n-1]$ where $y[n-1] = \begin{cases} n-1+1 & 0 \le n-1 \le 3 \\ 7-(n-1) & 4 \le n-1 \le 6 \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow y[n-1] = \begin{cases} n & 1 \le n \le 4 \\ 8-n & 5 \le n \le 7 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow h[n] = y[n] - y[n-1] = \begin{cases} n+1 & n = 0 \\ 1 & 1 \le n \le 3 \\ 7-2n & n = 4 \\ -1 & 5 \le n \le 6 \\ n-8 & n = 7 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & 0 \le n \le 3 \\ -1 & 4 \le n \le 7 \\ 0 & \text{otherwise} \end{cases}$$

$h[n]$



Question 4 Consider a periodic CT signal

$$x(t) = \begin{cases} \cos\left(\frac{\pi t}{4}\right) & -2 < t < 2 \\ \text{periodic with period } T = 4 \end{cases}$$

Q4.1 If we use inspection to have

$$x(t) = \frac{1}{2}\left(e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t}\right) = \frac{1}{2}\left(e^{j\frac{2\pi}{4}\left(\frac{1}{2}\right)t} + e^{j\frac{2\pi}{4}\left(-\frac{1}{2}\right)t}\right)$$

Since the Fourier series a_k should have integer k, this doesn't work

Therefore we use direct computation

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\frac{2\pi}{T}t}\,dt = \frac{1}{4}\int_{-2}^{2} \cos\left(\frac{\pi t}{4}\right) e^{-jk\frac{2\pi}{4}t}\,dt$$

$$= \frac{1}{4}\int_{-2}^{2} \frac{e^{j\frac{\pi}{4}t} + e^{-j\frac{\pi}{4}t}}{2}\, e^{-jk\frac{2\pi}{4}t}\,dt = \frac{1}{8}\int_{-2}^{2} e^{j(1-2k)\frac{\pi}{4}t} + e^{-j(1+2k)\frac{\pi}{4}t}\,dt$$

$$= \frac{1}{8}\left(\frac{e^{j(1-2k)\frac{\pi}{4}t}}{j(1-2k)\frac{\pi}{4}} - \frac{e^{-j(1+2k)\frac{\pi}{4}t}}{j(1+2k)\frac{\pi}{4}} \right)\Bigg|_{-2}^{2}$$

$$\boxed{= \frac{1}{2}\left(\frac{e^{j(1-2k)\frac{\pi}{2}} - e^{-j(1-2k)\frac{\pi}{2}}}{j(1-2k)\pi} - \frac{e^{-j(1+2k)\frac{\pi}{2}} - e^{j(1+2k)\frac{\pi}{2}}}{j(1+2k)\pi} \right)}$$

You will receive full credit if your answer is equal to the one on the left. The additional simplification is optional.

$$= \frac{\sin\left(\frac{\pi}{2}(1-2k)\right)}{(1-2k)\pi} + \frac{\sin\left(\frac{\pi}{2}(1+2k)\right)}{(1+2k)\pi}$$

If k is even $k = 2m$ $m \in \mathbb{Z}$

$$a_k = \frac{\sin\left(\frac{\pi}{2}(1-4m)\right)}{(1-2k)\pi} + \frac{\sin\left(\frac{\pi}{2}(1+4m)\right)}{(1+2k)\pi} = \frac{\sin\left(\frac{\pi}{2} - 2m\pi\right)}{(1-2k)\pi} + \frac{\sin\left(\frac{\pi}{2} + 2m\pi\right)}{(1+2k)\pi}$$

$$= \frac{\sin\left(\frac{\pi}{2}\right)}{(1-2k)\pi} + \frac{\sin\left(\frac{\pi}{2}\right)}{(1+2k)\pi} = \frac{1}{(1-2k)\pi} + \frac{1}{(1+2k)\pi} = \frac{-2}{(4k^2-1)\pi}$$

If k is odd $k = 2m+1$ $m \in \mathbb{Z}$

$$a_k = \frac{\sin\left(\frac{\pi}{2}(-1-4m)\right)}{(1-2k)\pi} + \frac{\sin\left(\frac{\pi}{2}(3+4m)\right)}{(1+2k)\pi} = \frac{\sin\left(-\frac{\pi}{2} - 2m\pi\right)}{(1-2k)\pi} + \frac{\sin\left(\frac{3\pi}{2} + 2m\pi\right)}{(1+2k)\pi}$$

$$= \frac{\sin\left(-\frac{\pi}{2}\right)}{(1-2k)\pi} + \frac{\sin\left(-\frac{\pi}{2}\right)}{(1+2k)\pi} = \frac{-1}{(1-2k)\pi} + \frac{-1}{(1+2k)\pi} = \frac{2}{(4k^2-1)\pi}$$

Therefore Fourier series coefficients

$$a_k = \frac{\sin\left(\frac{\pi}{2}(1-2k)\right)}{(1-2k)\pi} + \frac{\sin\left(\frac{\pi}{2}(1+2k)\right)}{(1+2k)\pi} = \begin{cases} \frac{-2}{(4k^2-1)\pi} & k \text{ even} \\ \frac{2}{(4k^2-1)\pi} & k \text{ odd} \end{cases}$$

Q 4.2 The Fourier series coefficients satisfying

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$ If we plug in $t=0$, we have

$$x(0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}\cdot 0} = \sum_{k=-\infty}^{\infty} a_k$$ Have $x(t) = \cos\left(\frac{\pi t}{4}\right)$ $-2 < t < 2$

Therefore $$\sum_{k=-\infty}^{\infty} a_k = \cos\left(\frac{\pi \cdot 0}{4}\right) = \cos(0) = 1$$

Question 5 Consider a DT periodic signal

$$x[n] = \sin\left(\frac{3\pi}{4}n + \frac{\pi}{4}\right)$$

Since $\frac{2\pi}{3\pi/4} = \frac{8}{3}$, the fundamental period of $x[n]$ is $N=8$

We can find Fourier series representation by inspection.

$$\begin{aligned} x[n] &= \frac{1}{2j}\left(e^{j(\frac{3\pi}{4}n+\frac{\pi}{4})} - e^{-j(\frac{3\pi}{4}n+\frac{\pi}{4})}\right) \\ &= \frac{e^{j\frac{\pi}{4}}}{2j} e^{j3\cdot\frac{2\pi}{8}n} - \frac{e^{-j\frac{\pi}{4}}}{2j} e^{j(-3)\frac{2\pi}{8}n} \\ &= \sum_{k=-4}^{3}\left(\frac{e^{j\frac{\pi}{4}}}{2j}\delta[k-3] - \frac{e^{-j\frac{\pi}{4}}}{2j}\delta[k+3]\right) e^{jk\frac{2\pi}{8}n} \end{aligned}$$

where the Fourier series coefficients

$$a_k = \begin{cases} \frac{e^{j\frac{\pi}{4}}}{2j} & k=3 \\ -\frac{e^{-j\frac{\pi}{4}}}{2j} & k=-3 \\ 0 & k=-4,-2,-1,0,1,2 \end{cases}$$

periodic with period $N=8$

Question 6

Consider the system with input $x_1(t)$ and output

$$y_1(t) = \begin{cases} \int_{-\infty}^{t+1} x_1(s-1)\, e^{t-s}\, ds & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

By changing variables, let $u = s-1$ and $du = ds$, we have the output

$$y_1(t) = \begin{cases} \int_{-\infty}^{t} x_1(u)\, e^{t-u-1}\, du & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

1° System 1 is not memoryless since $y_1(t)$ is the integral from time $-\infty$ to time t for $t \geq 0$ $\Rightarrow$ $y_1(t)$ depends on $x_1(u)$ $u \leq t$

2° Since $y_1(t)$ depends on only $x_1(u)$ $u \leq t$, system 1 is causal

3° Consider $x_1(t) = e^t u(-t) = \begin{cases} e^t & t \leq 0 \\ 0 & t > 0 \end{cases}$ as the input

Then $y_1(t) = \begin{cases} \int_{-\infty}^{t} e^u U(-u)\, e^{t-u-1}\, du & t \geq 0 \\ 0 & t < 0 \end{cases} = \begin{cases} e^{t-1}\int_{-\infty}^{0} 1\, du & t \geq 0 \\ 0 & t < 0 \end{cases}$

It is clear that $|x_1(t)| \le e^0 = 1$ for all t $\Rightarrow$ $x_1(t)$ is bounded

However $y_1(t) = e^{t-1} \int_{-\infty}^{0} 1\, du \to \infty$ for $t \ge 0$

$\Rightarrow$ $y_1(t)$ is not bounded $\Rightarrow$ System 1 is unstable

4° Consider two inputs $x_a(t)$ and $x_b(t)$ for system 1, we have outputs

$$y_a(t) = \begin{cases} \int_{-\infty}^{t} x_a(u) e^{t-u-1} du & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$y_b(t) = \begin{cases} \int_{-\infty}^{t} x_b(u) e^{t-u-1} du & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Therefore for input $x_1(t) = a\, x_a(t) + b\, x_b(t)$, the output is

$$y_1(t) = \begin{cases} \int_{-\infty}^{t} a x_a(u) + b x_b(u)\, e^{t-u-1} du & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$= \begin{cases} a \int_{-\infty}^{t} x_a(u) e^{t-u-1} du + b \int_{-\infty}^{t} x_b(u) e^{t-u-1} du & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$= a\, y_a(t) + b\, y_b(t)$ $\Rightarrow$ System 1 is linear

5° Consider the input $x_1'(t) = x_1(t - t_0)$, we have

$\int_{-\infty}^{t} x_1'(u) e^{t-u-1} du = \int_{-\infty}^{t} x_1(u - t_0) e^{t-u-1} du$ Let $v = u - t_0$ $dv = du$

$= \int_{-\infty}^{t-t_0} x_1(v) e^{t-t_0-v-1} dv$ Therefore we have output

$$y_1'(t) = \begin{cases} \int_{-\infty}^{t-t_0} x_1(v) e^{t-t_0-v-1} dv & t \ge 0 \\ 0 & t < 0 \end{cases}$$ where

$$y_1(t-t_0) = \begin{cases} \int_{-\infty}^{t-t_0} x_1(v) e^{t-t_0-v-1} dv & t - t_0 \ge 0 \\ 0 & t - t_0 < 0 \end{cases} \Rightarrow y_1'(t) \neq y_1(t - t_0)$$

$\Rightarrow$ System 1 is time variant

Consider the system 2 with input $x_2[n]$ and output

$y_2[n] = y_2[n-1] + \cos(3n) \cdot x_2[n+1] \cdot u[n-100]$

We first express $y_2[n]$ explicitly without the difference equation

Since $y_2[-200]=0$

$$y_2[\cancel{-199}] = y_2[-200] + \cos(3\cdot(-199))\cdot x_2(-198]\cdot U[-199-100] \qquad n=-199$$

$$y_2[\cancel{-198}] = y_2[\cancel{-199}] + \cos(3n)\cdot x_2[n+1]\cdot U[n-100) \qquad n=-198$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_2(n) = \cancel{y_2[n-1]} + \cos(3n)\cdot x_2[n+1]\cdot U(n-100] \qquad \text{sum from } k=-199 \text{ to } n$$

$$y_2[n] = y_2[-200] + \sum_{k=-199}^{n}\cos(3k)\cdot x_2[k+1]\cdot U[k-100]$$

$$\Rightarrow \quad y_2[n] = \sum_{k=-199}^{n}\cos(3k)\cdot x_2(k+1]\cdot U[k-100] = \begin{cases}\sum_{k=100}^{n}\cos(3k)\cdot x_2[k+1] & n\geq 100\\ 0 & -200\leq n<100\end{cases}$$

1° System 2 is not memoryless since $y_2[n]$ depends on $x_2[k]$ $101\leq k\leq n+1$

2° System 2 is not causal since $y_2[n]$ depends on $x_2[n+1]$

3° Consider $x_2[n] = \begin{cases}1 & \text{if } \cos(3n-3)\geq 0\\ -1 & \text{if } \cos(3n-3)<0\end{cases} \Rightarrow x_2[n+1] = \begin{cases}1 & \text{if } \cos(3n)\geq 0\\ -1 & \text{if } \cos(3n)<0\end{cases}$

Then $y_2[n] = \begin{cases}\sum_{k=100}^{n}\cos(3k)\cdot x_2[k+1] = \sum_{k=100}^{n}|\cos(3k)| & n\geq 100\\ 0 & -200\leq n<100\end{cases}$

It is clear that $|x_2[n]|=1$ for all n $\Rightarrow$ $x_2[n]$ is bounded

However when $n\to\infty$ $\quad y_2[n] = \lim_{n\to\infty}\sum_{k=100}^{n}|\cos(3k)| \to\infty$ is not bounded

$\Rightarrow$ System 2 is unstable

4° Consider two inputs $x_a[n]$ and $x_b[n]$ for system 2, we have outputs

$$y_a[n] = \begin{cases}\sum_{k=100}^{n}\cos(3k)\cdot x_a[k+1] & n\geq 100\\ 0 & -200\leq n<100\end{cases}$$

$$y_b(n) = \begin{cases}\sum_{k=100}^{n}\cos(3k)\cdot x_b[k+1] & n\geq 100\\ 0 & -200\leq n<100\end{cases}$$

Therefore for input $x_2[n] = a\,x_a(n) + b\,x_b(n]$, the output is

$$y_2[n) = \begin{cases}\sum_{k=100}^{n}\cos(3k)\left(a x_a(n) + b x_b[n]\right) = a\sum_{k=100}^{n}\cos(3k)\,x_a(n) + b\sum_{k=100}^{n}\cos(3k)x_b(n] & n\geq 100\\ 0 & -200\leq n<100\end{cases}$$

$$= a\,y_a[n] + b\,y_b(n)$$

$\Rightarrow$ System 2 is linear

5° Consider the input $x_2'[n] = x_2[n-n_0]$, we have

$$y_2'[n] = \begin{cases} \sum_{k=100}^{n} \cos(3k)\, x_2'[n] = \sum_{k=100}^{n} \cos(3k)\, x_2[k-n_0+1] & n \ge 100 \\ 0 & -200 \le n < 100 \end{cases}$$ let $m = k - n_0$

$$= \begin{cases} \sum_{m=100-n_0}^{n-n_0} \cos(3m+3n_0)\, x_2[m+1] & n \ge 100 \\ 0 & -200 \le n < 100 \end{cases}$$

$$y_2[n-n_0] = \begin{cases} \sum_{m=100}^{n-n_0} \cos(3m)\, x_2[m+1] & n-n_0 \ge 100 \\ 0 & -200 \le n-n_0 < 100 \end{cases}$$

Since $y_2'[n] \neq y_2[n-n_0]$ $\Rightarrow$ System 2 is time variant