Midterm #2 of ECE 301-002, 003, 004, 005 (CRN: 17727, 17101, 17102, 25618)
8–9pm, Wednesday, October 14, 2020, Online Exam.

1. Enter your student ID number, and signature in the space provided on this page **now!**
2. This is a closed book exam.
3. This exam contains multiple-choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and first working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed. No earphones are allowed either.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [23%, Work-out question] Consider the following DT Linear system

$$y[n] = \sum_{k=-|n|}^{2-|n|} x[k+2] \cdot |k| \tag{1}$$

1. [13%] Denote the *impulse response* of this system by $h[n]$. Plot $h[n]$ for the range of $-6 \le n \le 6$.

   Hint: There is no need to find the mathematical expression of $h[n]$. We only ask you to plot $h[n]$.

2. [4%] Consider a new signal $x[n] = U[n] - U[n-2]$ as the input to this LTI system and denote the corresponding output by $y[n]$. Is the following statement true or false?

   "Statement: The equality $y[n] = x[n] * h[n]$ holds."

   Please carefully explain your answer/reason. A yes/no answer without any justification will receive zero credit.

3. [6%] Please explain why this system is NOT invertible. You need to carefully justify your answer(s).

   Hint 1: If you do not know how to prove that this system is not invertible, please write down (i) the definition of an invertible system, and (ii) what is the general plan of proving a system is not invertible even if you do not know how to carry out the plan. You will receive 4 points (2 for each) if your answers to (i) and (ii) are correct.

   Hint 2: This may be among the harder questions of this exam. You may consider come back later after you finish the rest of the exam.

Question 2: [14%, Work-out question, Learning Objectives 1, 2, and 3] Consider an LTI system with the impulse response being

$$h(t) = \delta(t - 4\sqrt{2}) + e^{-t}U(t+1) \tag{2}$$

1. [14%] What is the output $y(t)$ when the input is $\cos(3t + 1.5)$?

   Hint 1: The following formulas may be useful:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{3}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{4}$$

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \quad \text{if } |r| < 1. \tag{5}$$

   Hint 2: You can write your answer to be of the form of $\frac{1}{4+j}e^{j5t}$. There is no need to further simplify your answers.

Question 3: [12%, Work-out question, Learning Objectives 2, 3, and 4]

Consider a DT-LTI system. We know that when the input is $x[n] = \mathcal{U}[n]$, the output is

$$y[n] = \begin{cases} n+1 & \text{if } 0 \leq n \leq 3 \\ 7-n & \text{if } 4 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Find the impulse response $h[n]$ of this DT-LTI system. You need to write down both the mathematical equations and plot $h[n]$ for the range of $-10 \leq n \leq 10$. If you only plot $h[n]$, you will receive 10 points.

Hint: If you do not know the answer to this question, please write down (i) [2%] the definition of *step response*; and (iii) [4%] Detailed steps about how to check whether a system is time-invariant or not. You will receive 2 points and 4 points respectively if your answers are correct.

Question 4: [20%, Work-out question, Learning Objectives 4 and 5] Consider a periodic CT signal

$$x(t) = \begin{cases} \cos\left(\frac{\pi \cdot t}{4}\right) & \text{if } -2 < t < 2 \\ \text{periodic with period } T = 4 \end{cases} \tag{7}$$

1. [15%] Find the Fourier series coefficients a_k of $x(t)$.

   Hint 1: There are two methods of finding CTFS. One is by inspection and one is by direct computation. And one of these two methods will NOT work. Please choose your method carefully.

   Hint 2:The following formula may be useful:

   $$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{8}$$

2. [5%] Find the value of $\sum_{k=-\infty}^{\infty} a_k$.

Question 5: [11%, Work-out question] Consider a DT periodic signal $x[n] = \sin(\frac{3\pi}{4}n + \frac{\pi}{4})$. Find the FS representation of $x[n]$, i.e., the corresponding (a_k, N).

Question 6: [20%, Multiple Choices, Learning Objective 1]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \begin{cases} \int_{-\infty}^{t+1} x(s-1)e^{t-s}ds & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{9}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = y_2[n-1] + \cos(3n) \cdot x_2[n+1] \cdot \mathcal{U}[n-100] \tag{10}$$

and we also known $y_2[-200] = 0$.

Answer the following questions

1. [4%, Learning Objective 1] Is System 1 memoryless? Is System 2 memoryless?
2. [4%, Learning Objective 1] Is System 1 causal? Is System 2 causal?
3. [4%, Learning Objective 1] Is System 1 stable? Is System 2 stable?
4. [4%, Learning Objective 1] Is System 1 linear? Is System 2 linear?
5. [4%, Learning Objective 1] Is System 1 time-invariant? Is System 2 time-invariant?

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2.$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n - r]$	Na_kb_k
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_lb_{k-l}$
First Difference	$x[n] - x[n - 1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$$