Question 1: [14%, Work-out question, Learning Objectives 4, 5] Consider the two following discrete-time signals:

$$x[n] = \cos(\frac{\pi n}{4}) \quad (1)$$

$$y[n] = \cos(\frac{5\pi n + \pi}{4}) \quad (2)$$

1. [2%] Find the DTFS a_k of $x[n]$.
2. [6%] Find the DTFS b_k of $y[n]$ and plot b_k for the range of $k = 0$ to 10.
3. [6%] Let $z[n] = x[n] \cdot y[n]$. Let c_k denote the DTFS of $z[n]$. Find the values of c_2.

   Hint: If you do not know the answers of the previous two sub-questions you can assume $a_k = 1$ if $0 \le k \le 2$ and $a_k = 0$ if $3 \le k \le 7$ and write c_2 as a function of b_k. You will receive 4 points if your answer is correct.

Answer:

1. Using the transfer pair in Table 5.2 with $N_1 = \frac{2\pi}{\frac{\pi}{4}} = 8$.

$$a_k = \begin{cases} \frac{1}{2} & k = \pm 1,\ \pm 1 \pm N_1,\ \pm 1 + 2N_1,\ \cdots \\ 0 & \text{otherwise} \end{cases}$$

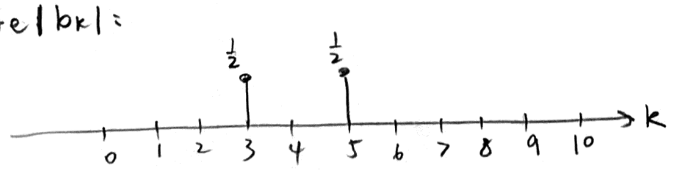
2. $$y[n] = \frac{1}{2} e^{j(\frac{5\pi n + \pi}{4})} + \frac{1}{2} e^{-j(\frac{5\pi n + \pi}{4})}$$

$$= \frac{1}{2} e^{j\frac{\pi}{4}} e^{j5\cdot(\frac{2\pi}{8})n} + \frac{1}{2} e^{-j\frac{\pi}{4}} e^{-j5\cdot(\frac{2\pi}{8})\cdot n}$$

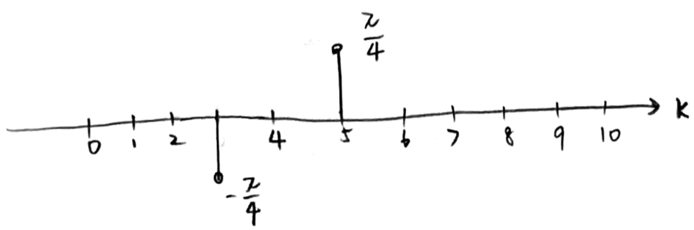
Thus $b_5 = \frac{1}{2} e^{j\frac{\pi}{4}}$

$b_{-5} = \frac{1}{2} e^{-j\frac{\pi}{4}}$ and it is periodic with $N_2 = 8$.

magnitute $|b_k|$:



phase $\angle b_k$:



3、 $z[n] = x[n] \cdot y[n]$

By multiplication property in Table 3.2

$$C_k = \sum_{l=\langle N \rangle} a_l b_{k-l}$$

Thus $C_2 = \sum_{l=\langle N \rangle} a_l b_{2-l} = a_1 b_1 + a_7 b_{-5}$

$$= \frac{1}{4} e^{-j\frac{\pi}{4}}.$$

(3) Alternative :

$$C_2 = \sum_{l=\langle N \rangle} a_l b_{2-l} = a_0 b_2 + a_1 b_1 + a_2 b_0$$

$$= b_2 + b_1 + b_0 .$$

Question 2: [15%, Work-out question, Learning Objectives 2, 3, 4, and 5] Consider the following continuous-time signals:

$$x(t) = \frac{\sin(3t)}{2t} \tag{3}$$

$$y(t) = \frac{\sin(2t)}{4t} \tag{4}$$

$$z(t) = (x(t) \cdot y(t)) * x(t). \tag{5}$$

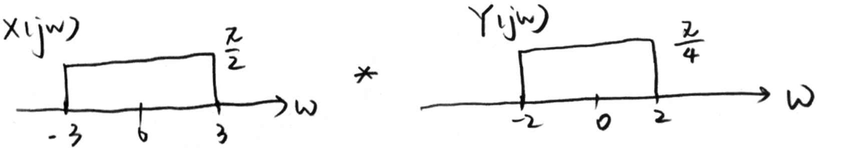
Find the CTFT $Z(j\omega)$ of $z(t)$.

Answer: let $c(t) = x(t) \cdot y(t)$

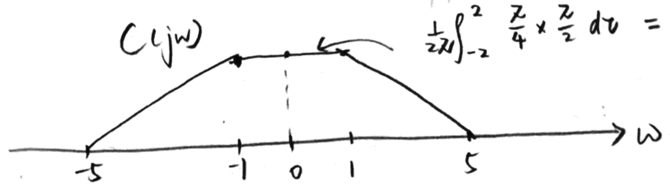
$\mathcal{F}\{x(t) \cdot y(t)\} = \frac{1}{2\pi} X(w) * Y(w)$ (multiplication property Table 4.1)

$x(t) = \frac{\sin(3t)}{\pi t} \cdot \frac{\pi}{2}$ $W_1 = 3$

$y(t) = \frac{\sin(2t)}{\pi t} \cdot \frac{\pi}{4}$ $W_2 = 2$



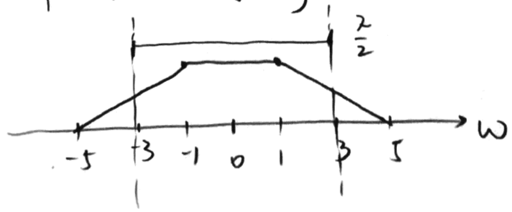
$\Rightarrow$



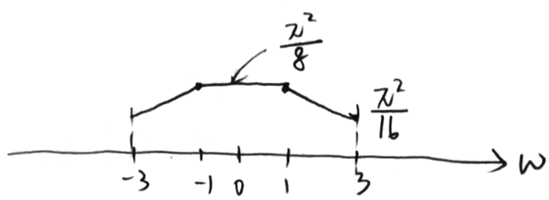
$\frac{1}{2\pi}\int_{-2}^{2} \frac{\pi}{4} \times \frac{\pi}{2} \, d\tau = \frac{\pi}{4}$

$z(t) = c(t) * x(t) = \mathcal{F}^{-1}\{C(jw) \cdot X(jw)\}$

$Z(jw) = C(jw) \cdot X(jw)$



$\Rightarrow$



Question 3: [10%, Work-out question, Learning Objectives 4 and 5] Consider a continuous-time LTI system governed by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 2\frac{dx(t)}{dt} + 3x(t) \tag{6}$$

1. [10%] Find the expression of $y(t)$ when the input is $x(t) = \sum_{k=5}^{7} e^{j(k^2-k)t}$.

$$j^2\omega^2 Y(j\omega) + 3j\omega Y(j\omega) + 2Y(j\omega) = 2j\omega X(j\omega) + 3X(j\omega)$$

$$\frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{2j\omega + 3}{-\omega^2 + 3j\omega + 2}$$

$$x(t) = e^{j20t} + e^{j30t} + e^{j42t}$$

$$y(t) = H(j20)e^{j20t} + H(j30)e^{j30t} + H(j42)e^{j42t}$$

$$y(t) = \frac{(3+j40)e^{j20t}}{-(20)^2 + 3j(20) + 2} + \frac{(3+j60)e^{j30t}}{-(30)^2 + 3j(30) + 2} + \frac{(3+j84)e^{j42t}}{-(42)^2 + 3j(42) + 2}$$

Question 4: [20%, Work-out question, Learning Objectives 2, 3, 4 and 5] Consider the following DT signal $x[n] = e^{j3(n-100)} \cdot 2^{-n+100} U[n-100]$.

1. [8%] Find the expression of $X(e^{j\omega})$.
2. [6%] Find the value of $\int_{\pi}^{3\pi} X(e^{j\omega}) d\omega$.
3. [6%] Find the value of $\int_{0}^{10\pi} |X(e^{j\omega})|^2 d\omega$.

Hint: The following formula may be useful: If $|r| < 1$, then

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}. \quad = \sum_{k=0}^{\infty} ar^k \tag{7}$$

1. $X(j\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$= \sum_{n=100}^{\infty} e^{j3(n-100)} 2^{-n+100} e^{-j\omega n}$$

$k = n - 100$

$n = k + 100$

$$\rightarrow \sum_{k=0}^{\infty} e^{j3k} 2^{-k} e^{-j\omega(k+100)}$$

$$= e^{-j\omega 100} \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{3j - j\omega}\right)^k$$

$$= \boxed{\frac{e^{-j\omega 100}}{1 - \frac{1}{2} e^{3j - j\omega}}}$$

periodic with period 2π

2. $x[n] = \frac{1}{2\pi}\int_{2\pi} X(j\omega)e^{j\omega n}\,d\omega$

$$\int_{\pi}^{3\pi} X(j\omega)\,d\omega = \int_{2\pi} X(j\omega)e^{j\omega 0}\,d\omega \quad n=0$$

$$= 2\pi\,x[0] = \boxed{0}$$

3. $$\int_{0}^{10\pi} |X(j\omega)|^2\,d\omega = 5\int_{2\pi} |X(j\omega)|^2\,d\omega$$

$$= 2\pi \cdot 5 \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad \text{(Parseval's Relation)}$$

$$= 10\pi \sum_{n=100}^{\infty} \cancel{|e^{j3(n-100)}|^2}^{\,1}\, |2^{-n+100}|^2$$

$$= 10\pi \sum_{n=100}^{\infty} 2^{-2n+200}$$

$k = n - 100$

$$\rightarrow\ 10\pi \sum_{k=0}^{\infty} 2^{-2(k+100)+200}$$

$$= 10\pi \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^n$$

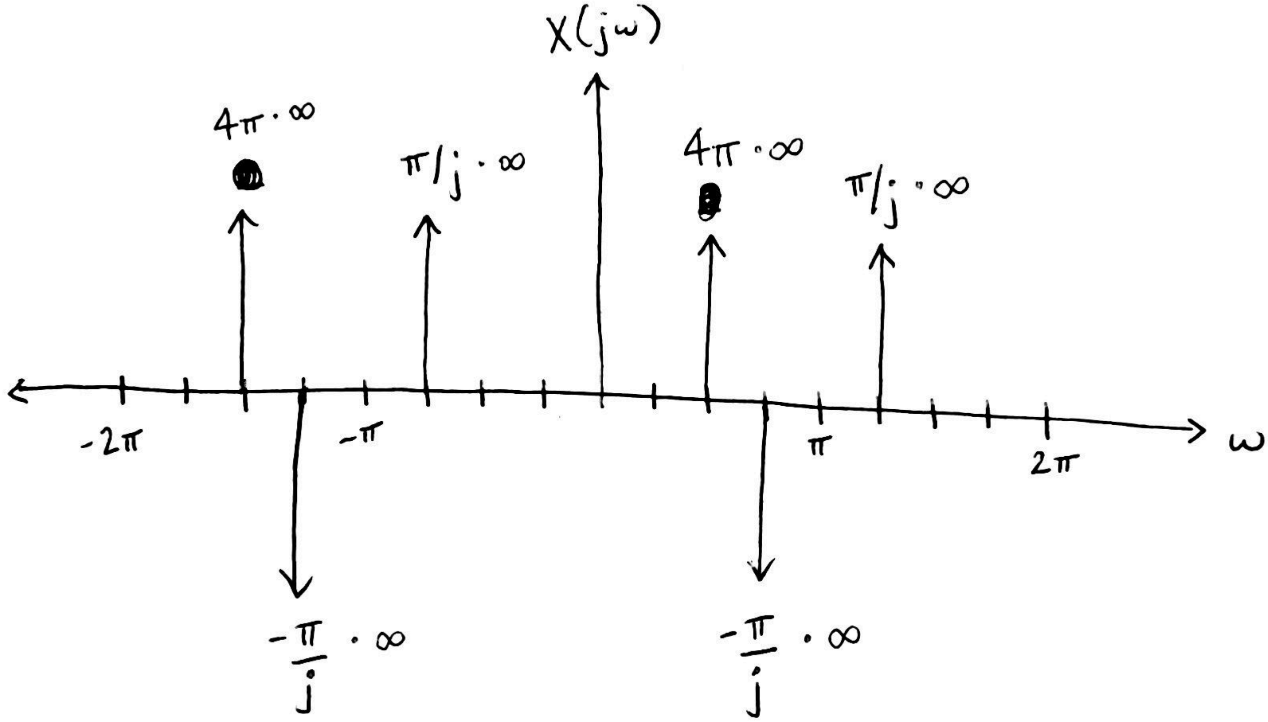
$$= \frac{10\pi}{1 - 1/4} = \boxed{\frac{40\pi}{3}}$$

Question 5: [18%, Work-out question, Learning Objectives 4, 5, and 6] Consider a discrete time signal

$$x[n] = \sin(1.25\pi n) + 2e^{j2.5\pi n} \tag{8}$$

Plot the corresponding DTFT $X(e^{j\omega})$ for the range of $-2\pi < \omega < 2\pi$.

Hint: There is no need to write down the expression of $X(e^{j\omega})$. A plot is sufficient.



Question 6: [23%, Learning Objectives 3, 4, 5, and 6] Consider the following *frequency scrambler* system. The input signal is $x(t) = \cos(2\pi t)$. We first multiply $x(t)$ by $\cos(10\pi t)$. That is,

$$y(t) = x(t) \cdot \cos(10\pi t).$$

We then pass $y(t)$ through a low pass filter with cutoff frequency 5Hz. Denote the final output by $z(t)$.

1. [4%] Find out the expression of the impulse response $h(t)$ for the low pass filter with cutoff frequency 5Hz.

2. [8%] Use the CTFT to carefully analyze the system and find the expression of $z(t)$. Please carefully write down how the frequency spectrums $X(j\omega)$, $Y(j\omega)$ and $Z(j\omega)$ evolve.

3. [11%] For the *descrambler*, a student figured out that all he/she needs to do is to multiple $z(t)$ by $\cos(10\pi t)$ again to get $w(t) = z(t) \cdot \cos(10\pi t)$ and then pass $w(t)$ through a low pass filter with cutoff frequency 5Hz. Denote the final output by $\hat{x}(t)$. However, when he/she played the descrambled signal $\hat{x}(t)$, he/she realized that $\hat{x}(t)$ is not identical to the original signal $x(t)$ anymore. Please (i) [7%] carefully analyze the descrambler in the frequency domain; (ii) [2%] Describe how $\hat{x}(t)$ sounds when compared to the original signal $x(t)$; and (iii) [2%] Describe how you will fix the descrambler so that $\hat{x}_{\text{new}}(t) = x(t)$.

Hint: If you do not know how to solve this question, you can write down the transmitter and receiver diagrams of AM-DSB when (i) filtering a radio signal to make it of bandwidth 7.5k Hz; (ii) then sending the filtered signal using carrier frequency 890kHz; and (iii) the receiver needs to demodulate the original signal. You need to carefully mark all the frequency parameters used in your scheme with the correct unit. If your answer is correct, you will receive 11 points.

Answer:

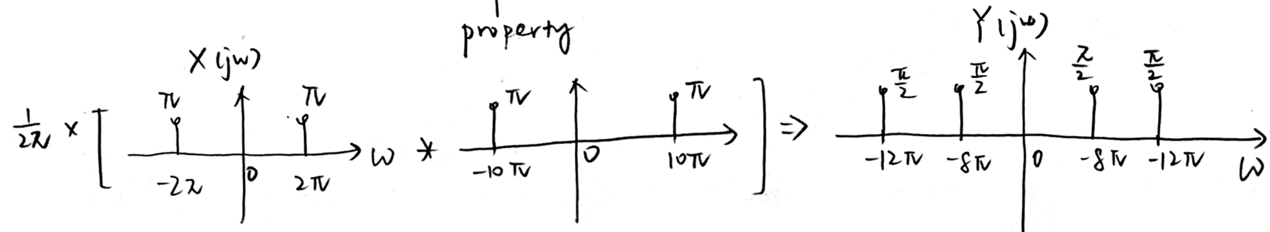
1. $f_{cutoff} = 5Hz$ $\quad W = 2\pi f_{cutoff} = 10\pi$ rad/s.

$$h(t) = \frac{\sin Wt}{\pi t}$$ (the sinc function)

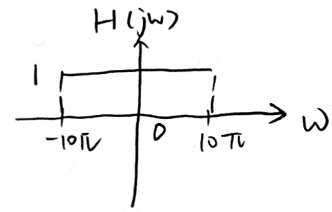
2. $$\begin{array}{ccccc} x(t) \longrightarrow & \otimes & \xrightarrow{y(t)} & \boxed{h(t)} & \longrightarrow z(t) \\ & \uparrow & & & \\ & \cos(10\pi t) & & & \end{array}$$

a. Modulate $x(t)$ with $\cos(10\pi t)$:

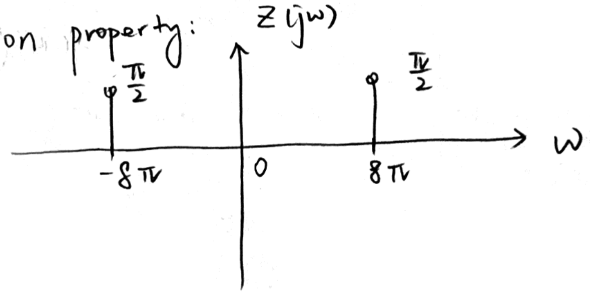
$$y(t) = x(t)\cos(10\pi t) \xrightarrow[\text{property}]{\text{Multiplication}} Y(j\omega) = \frac{1}{2\pi}\cdot X(j\omega) * [\pi\delta(\omega-10\pi) + \pi\delta(\omega+10\pi)]$$



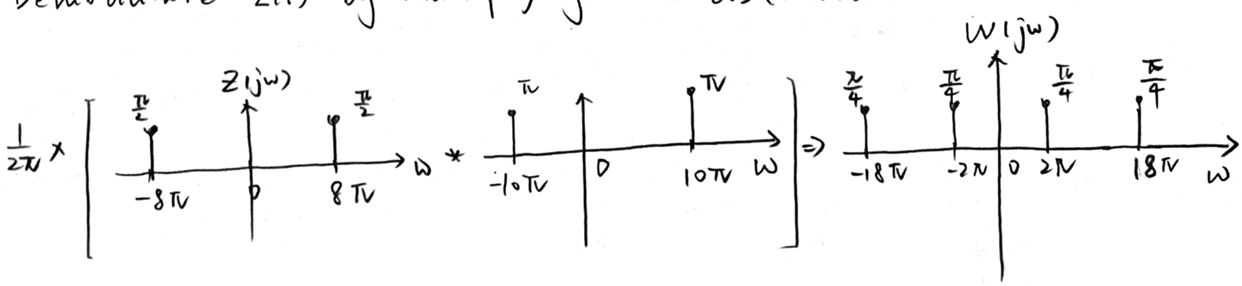
b. Pass $y(t)$ through the low pass filter $h(t)$:



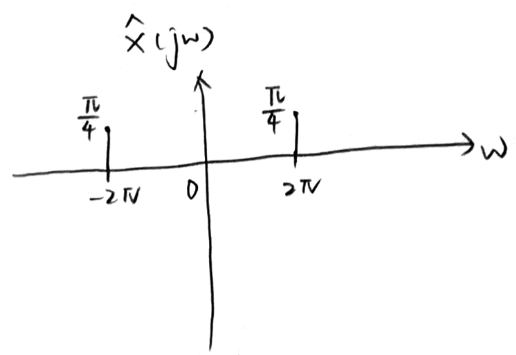
By convolution property:



3. a. Demodulate $z(t)$ by multiplying with $\cos(10\pi t)$:



b. Pass $w(t)$ through $h(t)$ (low pass):

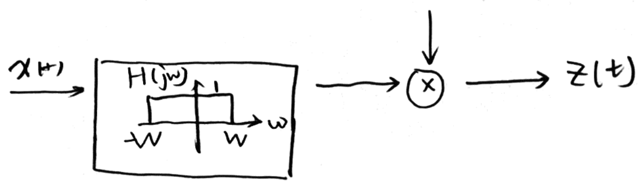


$\hat{x}(t)$ has a lower volumn than $x(t)$. We can make the low pass filter used for descrambler, $h_2(t) = 4h(t)$.

Hint question:

Transmitter:

$\cos w_0 t$ $\quad w_0 = 2\pi f = 2\pi \times 890 \times 10^3 = 1.780\pi \times 10^6$ rad/s



$W = 2\pi f_{cutoff} = 15\pi \times 10^3$ rad/s

receiver: