Question 1: [20%, Work-out question, Learning Objectives 1, 2, and 3]

1. [2%] What does the acronym *LTI* stand for?

2. [14%] Consider a DT-LTI system with the impulse response being

$$h[n] = \begin{cases} n+1 & \text{if } -1 \leq n \leq 2 \\ 5-n & \text{if } 3 \leq n \leq 5 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

   Suppose we use $x[n] = U[n+1] - U[n] + U[n-2] - U[n-3]$ as the input to this LTI system and denote the corresponding output by $y[n]$. Plot $y[n]$ for the range of $-8 \leq n \leq 8$.

   Hint 1: There is no need to find the mathematical expression of $y[n]$. We only ask you to plot $y[n]$.

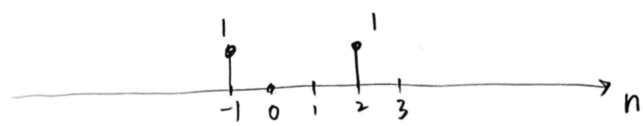
   Hint 2: It may be useful to plot $x[n]$ first.

3. [4%] Is the above system causal? Please carefully write down your reasons. A yes/no answer without reasons will not receive any point.

Answer

1. LTI : linear time - invariant.

2. $x[n]$:



$x[-k]$:

$h[k]$:

$x[n] * h[n]$

$= \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$

Alternatively:

$h[n] * x[n]$

$= \sum_{k=-\infty}^{+\infty} h[n-k]x[k] \quad \left(x[k] = \delta[k+1] + \delta[k-2]\right)$

$= h[n+1] + h[n-2]$

3. It is causal. Because $h[n]=0$ for all $n<0$. An input $\delta[n]$ only affects the outputs when $n \geq 0$. (In the present or future).

Question 2: [15%, Work-out question, Learning Objectives 1, 2, and 3] Consider the following two signals:

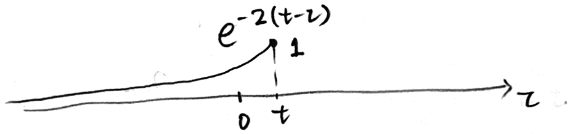
$$x_1(t) = e^{-2t}U(t) \tag{2}$$
$$x_2(t) = e^{-jt}U(t-\pi) \tag{3}$$

1. Compute the convolution $y(t) = x_1(t) * x_2(t)$.

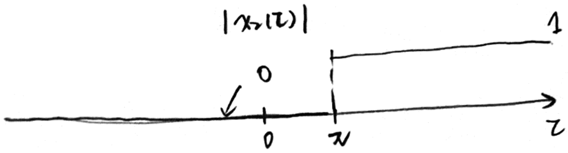
   Hint 1: There is no need to greatly simplify your answers. Some expressions of the form of $\frac{3}{3+4j}e^{j\sqrt{2}t}$ would suffice.

Answer: $y(t) = x_1(t) * x_2(t) = \int_{\tau=-\infty}^{\infty} x_1(t-\tau)\cdot x_2(\tau)\, d\tau$

$x_1(t-\tau)$:



$x_2(\tau)$:



Case 1: when $t \le \pi$, no overlapping.

$y(t) = 0, \quad t \le \pi$

Case 2: when $t > \pi$ overlapping on $[\pi, t]$.

$$y(t) = \int_{\tau=\pi}^{t} e^{-2(t-\tau)} \cdot e^{-j\tau}\, d\tau$$

$$= e^{-2t} \int_{\tau=\pi}^{t} e^{(2-j)\tau}\, d\tau$$

$$= e^{-2t} \cdot \left. \frac{e^{(2-j)\tau}}{2-j} \right|_{\tau=\pi}^{t}$$

$$= e^{-2t} \cdot \frac{e^{(2-j)t}}{2-j} - e^{-2t} \cdot \frac{e^{(2-j)\pi}}{2-j}$$

$$= \frac{e^{-jt}}{2-j} - \frac{e^{(2-j)\pi} \cdot e^{-2t}}{2-j}, \quad \text{for } t > \pi.$$

$$= \frac{e^{-jt}}{2-j} + \frac{e^{2\pi-2t}}{2-j}$$

Question 3: [18%, Work-out question, Learning Objectives 2, 3, and 4]
Consider a DT-LTI system with impulse response

$$h[n] = \begin{cases} e^{(-0.5+3j)n} & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Find the output $y[n]$ when the input is $x[n] = \cos(3n + \pi/3)$.

Hint 1: There is no need to greatly simplify your answers. Some expressions of the form of $\frac{3}{3+4j}e^{j\sqrt{2}t} + \frac{8}{3-2j}e^{j\sqrt{3}t}$ would suffice.

Hint 2: The following formulas may be useful:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{5}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{6}$$

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \quad \text{if } |r| < 1. \tag{7}$$

$$x[n] = \cos(3n + \pi/3) = \cos\left(3\left(n + \frac{\pi}{9}\right)\right)$$

$$= \frac{e^{j(3n+\frac{\pi}{3})} + e^{-j(3n+\pi/3)}}{2} = \underbrace{\frac{1}{2}e^{j(3n+\frac{\pi}{3})}}_{\omega = 3} + \underbrace{\frac{1}{2}e^{-j(3n+\frac{\pi}{3})}}_{\omega = -3}$$

$$H(j\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} e^{(-0.5+3j)n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(e^{-0.5+3j-j\omega}\right)^n \qquad \begin{pmatrix} l = n+1 \\ n = l-1 \end{pmatrix}$$

$$= \sum_{l=1}^{\infty} \left(e^{-0.5+3j-j\omega}\right)^{l-1} = \frac{1}{1 - e^{-0.5+3j-j\omega}}$$

$$H(j3) = \frac{1}{1 - e^{-0.5}} \qquad H(j(-3)) = \frac{1}{1 - e^{-0.5+6j}}$$

$$y[n] = H(j3)\left(\frac{1}{2}e^{j(3n+\frac{\pi}{3})}\right) + H(j(-3))\left(\frac{1}{2}e^{-j(3n+\frac{\pi}{3})}\right)$$

$$\boxed{y[n] = \frac{1}{2-2e^{-0.5}}e^{j(3n+\frac{\pi}{3})} + \frac{1}{2-2e^{-0.5+6j}}e^{-j(3n+\frac{\pi}{3})}}$$

Question 4: [19%, Work-out question, Learning Objectives 4 and 5]

1. [14%] Consider a periodic CT signal

$$x(t) = \begin{cases} \pi & \text{if } 0 < t < 2 \\ 0 & \text{if } 2 < t < 5 \\ \text{periodic with period } T = 5 & \end{cases} \quad (8)$$

Find the Fourier series coefficients a_k of $x(t)$.

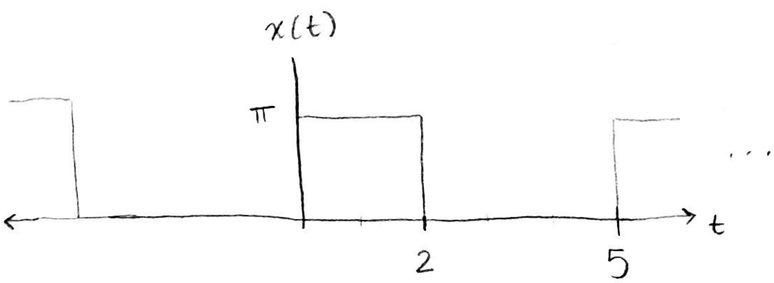
2. [5%] Find the values of $\sum_{k=-\infty}^{\infty} |a_k|^2$.

Hint 1: If you do not know the answer of the previous subquestion, you can assume that

$$a_k = \begin{cases} \frac{\sin(0.7k)}{k} & \text{if } k \neq 0 \\ 0.7 & \text{if } k = 0. \end{cases} \quad (9)$$

You will receive full credit if your answer is correct.

Hint 2: You may need to use the tables in the end of the exam booklet.



1. $a_k = \frac{1}{5}\int_0^5 x(t)\, e^{-jk\left(\frac{2\pi}{5}\right)t}\, dt$

$= \frac{1}{5}\int_0^2 \pi\, e^{-jk\left(\frac{2\pi}{5}\right)t}\, dt$

$= \frac{\pi}{5}\left[-\frac{1}{jk\left(\frac{2\pi}{5}\right)} e^{-jk\left(\frac{2\pi}{5}\right)t}\right]_{t=0}^{t=2}$

$= \frac{1}{2jk}\left[e^{-jk\left(\frac{2\pi}{5}\right)0} - e^{-jk\left(\frac{2\pi}{5}\right)2}\right]$

$$\boxed{a_k = \frac{1}{2jk} - \frac{e^{-jk\frac{4\pi}{5}}}{2jk}}$$

$$a_0 = \frac{1}{5}\int_0^2 \pi \, dt$$

$$\boxed{a_0 = \frac{2\pi}{5}}$$

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$$a_k = \frac{e^{-jk\frac{2\pi}{5}}}{2jk}\left[e^{jk\left(\frac{2\pi}{5}\right)} - e^{-jk\left(\frac{2\pi}{5}\right)}\right] \qquad \left(\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}\right)$$

$$a_k = \boxed{\frac{e^{-jk\frac{2\pi}{5}}\sin\left(\frac{2\pi}{5}k\right)}{k}} = \left(\frac{\sin\left(\frac{2\pi}{5}k\right)}{\pi k}\right) \cdot \underbrace{\pi}_{\text{amplitude scale}} \cdot \underbrace{e^{-jk\frac{2\pi}{5}}}_{\text{time shift}}$$

$$\left(T_1 = 1 \quad T = 5\right)$$

$$a_0 = \frac{2T_1}{T} \cdot \pi = \boxed{\frac{2\pi}{5}}$$

2. $$\sum_{k=-\infty}^{\infty} |a_k|^2 = \frac{1}{T}\int_T |x(t)|^2 dt \quad \text{(Parseval)}$$

$$= \frac{1}{5}\int_0^2 |\pi|^2 dt$$

$$= \boxed{\frac{2\pi^2}{5}}$$

Question 5: [8%, Work-out question, Learning Objective 4] Consider a DT periodic signal of period $N = 8$. Suppose we also know that its FS coefficients are $a_1 = 1 + j$, $a_7 = 1 - j$, and $a_k = 0$ for $k = 0, 2, 3, 4, 5, 6$.

Find the expression of $x[n]$.

Hint: In this question, $x[n]$ is a real-valued signal. Your answer cannot have the imaginary number j in the expression. You need to simplify the expression so that there is no j inside.

$$x[n] = \sum_{K=\langle N\rangle} a_K e^{jK\left(\frac{2\pi}{N}\right)n}$$

$$a_1 = 1 + j = \sqrt{2}\, e^{j\pi/4}$$

$$a_7 = 1 - j = \sqrt{2}\, e^{-j\pi/4}$$

$$N = 8$$

$$a_7 = a_{-1}$$

$$x[n] = \sqrt{2}\, e^{j\pi/4} e^{j\left(\frac{2\pi}{8}\right)n} + \sqrt{2}\, e^{-j\pi/4} e^{-j\left(\frac{2\pi}{8}\right)n}$$

$$x[n] = \sqrt{2}\, e^{j\frac{\pi}{4}n + \frac{\pi}{4}} + \sqrt{2}\, e^{-j\frac{\pi}{4}n - \frac{\pi}{4}}$$

$$\boxed{x[n] = 2\sqrt{2}\cos\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)}$$

Alternate:

$$x[n] = (1+j)\,e^{j\frac{\pi}{4}n} + (1-j)\,e^{-j\frac{\pi}{4}n} \qquad \left(j = -\frac{1}{j}\right)$$

$$= \left(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right) - \left(\frac{1}{j}e^{j\frac{\pi}{4}n} - \frac{1}{j}e^{-j\frac{\pi}{4}n}\right)$$

$$\boxed{x[n] = 2\cos\left(\frac{\pi}{4}n\right) - 2\sin\left(\frac{\pi}{4}n\right)}$$

Question 6: [20%, Multiple Choices, Learning Objective 1]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \begin{cases} \frac{e^{jx_1(t-1)}+e^{-jx_1(t-1)}}{e^{jx_1(t-1)}-e^{-jx_1(t-1)}} & \text{if } x_1(t-1) \neq 0 \\ 0 & \text{if } x_1(t-1) = 0 \end{cases} \tag{10}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = \int_{s=-\infty}^{\max(x[n],0)} e^s ds \tag{11}$$

Answer the following questions

1. [4%, Learning Objective 1] Is System 1 memoryless? Is System 2 memoryless?
2. [4%, Learning Objective 1] Is System 1 causal? Is System 2 causal?
3. [4%, Learning Objective 1] Is System 1 stable? Is System 2 stable?
4. [4%, Learning Objective 1] Is System 1 linear? Is System 2 linear?
5. [4%, Learning Objective 1] Is System 1 time-invariant? Is System 2 time-invariant?

System 1: $y_1(t) = \begin{cases} \frac{\cos(x_1(t-1))}{j\sin(x_1(t-1))} & \text{if } x_1(t-1) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$

System 1 is not memoryless, because $y_1(t)$ depends on $x_1(t-1)$;

causal, because $y_1(t)$ only depends on the past;

unstable, because $\cot(x_1(t-1)) = \frac{\cos(x_1(t-1))}{\sin(x_1(t-1))}$ can go to infinite with a finite $x_1(t-1)$;

not linear, $\frac{\cos(x_1(t-1)+x_2(t-1))}{\sin(x_1(t-1)+x_2(t-1))} \neq \frac{\cos x_1(t-1)}{\sin x_1(t-1)} + \frac{\cos(x_2(t-1))}{\sin(x_2(t-1))}$

time-invariant, $y_1(t-t_0) = \begin{cases} \frac{\cos(x_1(t-t_0-1))}{j\sin(x_1(t-t_0-1))} & \text{if } x_1(t-t_0-1) \neq \\ 0 & \text{otherwise.} \end{cases}$

$x_1(t-t_0) \rightarrow \square \rightarrow y_1'(t) = \begin{cases} \frac{\cos(x_1(t-t_0-1))}{j\sin(x_1(t-t_0-1))} & \text{if } x_1(t-t_0-1) \neq 0 \\ 0 & \text{otherwise} \end{cases}$

System 2:

$$y_2[n] = \int_{s=-\infty}^{\max(x_2[n],0)} e^s \, ds .$$

The sysem is <u>memoryless</u>, because $y_2[n]$ only depends on $x_2[n]$.

<u>Causal</u>, because $y_2[n]$ does not depend on future inputs.

<u>Stable</u>, given a bounded input $x_2[n]$, the integral is also bounded.

<u>Not linear</u>, example: $x_{21}[0] = 3$
$x_{22}[0] = -3$

$$y_2[0] = \int_{s=-\infty}^{\max(x_{21}[0]+x_{22}[0],\,0)} e^s \, ds$$

$$= \int_{s=-\infty}^{0} e^s \, ds$$

$$\neq$$

$$y_{21}[0] + y_{22}[0] = \int_{s=-\infty}^{3} e^s ds + \int_{s=-\infty}^{0} e^s ds$$

<u>time-invariant</u>, $y_2[n-n_0] = \int_{s=-\infty}^{\max[x_2[n-n_0],0)} e^s ds$

$$\|$$

$$x_2[n-n_0] \rightarrow \boxed{\ } \rightarrow y_2'[n] = \int_{s=-\infty}^{\max[x_2[n-n_0],0)} e^s ds .$$