Midterm #1 of ECE301-003, 004 (CRN 17101, 17102)
8–9pm, Thursday, September 12, 2019, LILY 1105.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: SOLUTION

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [20%, Work-out question, Learning Objectives 1 and 2] Consider two DT signals:

$$x_1[n] = (U[n+2] - U[n-4]) \cdot n^2 + 3 \cdot \delta[n+3] \cdot n \tag{1}$$
$$x_2(t) = U(-t+2) \cdot e^{jt^2} \cdot (\sin(t) - j\cos(t)) \cdot 2^t. \tag{2}$$

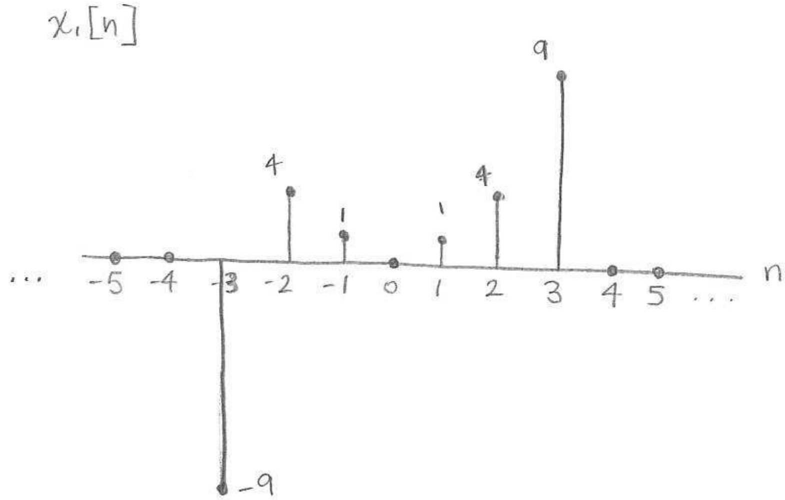
Plot

1. [5%] Plot $x_1[n]$ for the range of $-5 \le n \le 5$.
2. [7%] Plot the even part of $x_1[n]$ for the range of $-5 \le n \le 5$.

   Hint: If you do not know how to solve this question, write down the formulas of the odd part of $x_1[n]$. You will receive 3 points if your answer is correct.
3. [8%] Compute the *total energy* of $x_2(t)$.

$x_1[n]$

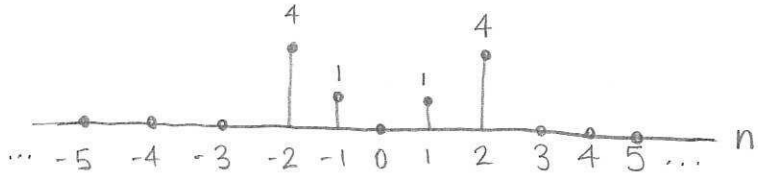
1.



$x_{1e}[n]$

$$x_{1e}[n] = \frac{x_1[n] + x_1[-n]}{2}$$

2.



n	$x_{1e}[n]$
0	0
1	$\frac{1+1}{2} = 1$
2	$\frac{4+4}{2} = 4$
3	$\frac{9-9}{2} = 0$

3. Total Energy $= \displaystyle\int_{-\infty}^{\infty} |x_2(t)|^2 \, dt$

$$= \int_{-\infty}^{2} \left| e^{jt^2} (\sin(t) - j\cos(t)) \, 2^t \right|^2 dt$$

$$= \int_{-\infty}^{2} |e^{jt^2}|^2 \, |e^{-jt}|^2 \, |2^t|^2 \, dt$$

always positive

$$= \int_{-\infty}^{2} (1) \cdot (1) \cdot 2^{2t} \, dt$$

$$= \left. \frac{4^t}{\ln(4)} \right|_{t=-\infty}^{t=2}$$

$$= \frac{16}{\ln(4)} - 0$$

$$= \boxed{\frac{16}{\ln(4)}}$$

Question 2: [16%, Work-out question, Learning Objective 3]

Define two DT signals:

$$x[n] = \begin{cases} 5^{(n-100)} & \text{if } n \leq 100 \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

and

$$h[n] = 3^{-|n-20|}. \tag{4}$$

Compute the expression of the following sum

$$z[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{5}$$

Hint 1: You can leave your answer to be of the form like $\frac{e^{3j}-2^{-12}}{3+10j}$. There is no need to further simplify the expression.

Hint 2: If $|r| < 1$, then we have the following formulas for computing the infinite sum of a geometric sequence.

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

$$\sum_{k=1}^{\infty} kar^{k-1} = \frac{a}{(1-r)^2}.$$

If $r \neq 1$, then we have the following formula for computing the finite sum of a geometric sequence.

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}.$$

Hint 3: **If you do not know how to carefully carry out the summation or if you do not have time to carry out the summation,** you can leave your answer to be something like:

$$z[n] = \begin{cases} \sum_{k=n-10}^{40} 2^{n-k} + \sum_{k=50}^{\infty} 2^{k-n} & \text{if } n \leq 50 \\ \sum_{k=0}^{\infty} 2^{k-n} & \text{otherwise} \end{cases} \tag{6}$$

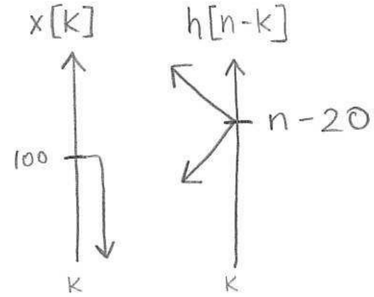
You will still receive 11 points if your answers are correct.

$$x[k] = \begin{cases} 5^{(k-100)}, & k \le 100 \\ 0, & \text{else} \end{cases}$$

$$h[n-k] = 3^{-|n-k-20|}$$

$$h[n-k] = \begin{cases} 3^{-n+k+20}, & n-k-20 \ge 0 \\ & k \le n-20 \\ 3^{n-k-20}, & n-k-20 < 0 \\ & k > n-20 \end{cases}$$

Case 1: $n - 20 \ge 100 \rightarrow n \ge 120$



$$z_1[n] = \sum_{k=-\infty}^{100} 5^{(k-100)} 3^{-n+k+20}$$

$$= 5^{-100} 3^{-n+20} \sum_{k=-\infty}^{100} 5^k 3^k$$

Let $\ell = -k + 101$

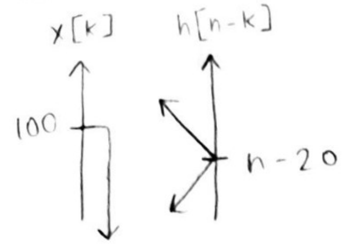
$$= 5^{-100} 3^{-n+20} \sum_{\ell=1}^{\infty} 15^{101-\ell}$$

$$= \cancel{5^{-100} 5^{100}}^{\rightarrow 1} 3^{100} 3^{-n+20} \sum_{\ell=1}^{\infty} \left(\frac{1}{15}\right)^{\ell-1}$$

$$= 3^{-n+120} \left(\frac{1}{1 - \frac{1}{15}}\right)$$

$$z_1[n] = 3^{-n+120} \left(\frac{3 \cdot 5}{14}\right) = \boxed{\frac{5 \cdot 3^{-n+121}}{14}}$$

Case 2 : $n-20 < 100 \longrightarrow n < 120$



$$z_2[n] = \sum_{k=-\infty}^{n-20} 5^{(k-100)} 3^{-n+k+20} + \sum_{k=n-19}^{100} 5^{(k-100)} 3^{n-k-20}$$

$$= 5^{-100}\left[3^{-n+20} \sum_{k=-\infty}^{n-20} 5^k 3^k + 3^{n-20} \sum_{k=n-19}^{100} 5^k 3^{-k} \right]$$

Let $\ell = -k+n-20+1$ ⋮ Let $m = k-n+19+1$

$$= 5^{-100}\left[3^{-n+20} \sum_{\ell=1}^{\infty} 15^{-\ell+n-20+1} + 3^{n-20} \sum_{m=1}^{120-n} \left(\frac{5}{3}\right)^{m+n-19-1} \right]$$

$$= 5^{-100}\left[\cancel{3^{-n+20}}\, 3^{n-20+1}\, 5^{n-20} \sum_{\ell=1}^{\infty} \left(\frac{1}{15}\right)^{\ell-1} + 3^{n-20}\, 5^{n-20}\, 3^{-n+19} \sum_{m=1}^{120-n} \left(\frac{5}{3}\right)^{m-1} \right]$$

$$= 5^{n-120}\left(\frac{1}{1-\frac{1}{15}}\right) + 3^{-1} 5^{n-120} \left(\frac{1-\left(\frac{5}{3}\right)^{120-n}}{1-\frac{5}{3}} \right)$$

$$= 5^{n-120}\left(\frac{3\cdot 5}{14}\right) + \left(3^{-1} 5^{n-120} - \cancel{5^{n-120}}\, \cancel{5^{120-n}}\, 3^{n-120}\, \cancel{3^{-1}} \right)\left(-\frac{\cancel{3}}{2}\right)$$

$$\boxed{z_2[n] = 5^{n-119}\left(\frac{3}{14}\right) + \frac{-5^{n-120} + 3^{n-120}}{2}}$$

$$z[n] = \begin{cases} z_1[n], & n \geq 120 \\ z_2[n], & n < 120 \end{cases}$$

Question 3: [22%, Work-out question, Learning Objectives 1, 4, and 5] Consider a CT signal

$$h(t) = \begin{cases} e^{(j-1)t} & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } -10 < t < 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Based on $h(t)$, we construct another signal

$$y(t) = \int_{s=-\infty}^{\infty} h(s) e^{j\omega(t-s)} ds \tag{8}$$

where we assume ω is an arbitrarily given constant.

1. [14%] Find the expression of $y(t)$ for arbitrarily given ω.

   Hint 1: Your answer of $y(t)$ should still have ω inside. Namely, for any different ω value, we will have a different $y(t)$. We are interested in a general formula of $y(t)$ for arbitrary ω.

2. [8%] Write down the expression of $y(t)$ when $\omega = 10\pi$ and carefully explain whether $y(t)$ (assuming $\omega = 10\pi$) is periodic or not. If it is periodic, please also write down the fundamental period.

   Hint 2: If you do not know the answer to the previous question, you can assume

   $$y(t) = \left(\frac{e^{j\omega t} + e^{-j\omega t}}{5} \right)^2 \tag{9}$$

   to solve this question. You may need to use the equalities

   $$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{10}$$
   $$e^{-j\theta} = \cos(\theta) - j\sin(\theta) \tag{11}$$
   $$\sin(\theta)\sin(\phi) = \frac{\cos(\theta - \phi) - \cos(\theta + \phi)}{2} \tag{12}$$

   along the process. You will receive full credits if your final answer is correct.

Q3. 22 points

1) $$y(t) = \int_{s=-\infty}^{\infty} h(s) e^{jw(t-s)} ds$$

$$= \int_{s=-10}^{0} e^{jw(t-s)} ds + \int_{s=0}^{1} e^{(j-1)s} e^{jw(t-s)} ds$$

$$= e^{jwt} \cdot \frac{e^{-jws}}{-jw} \Bigg|_{s=-10}^{0} + e^{jwt} \cdot \frac{e^{(j-jw-1)s}}{j-jw-1} \Bigg|_{s=0}^{1}$$

$$= e^{jwt} \left[\frac{1}{-jw} - \frac{e^{j10w}}{-jw} + \frac{e^{j-jw-1}}{j-jw-1} - \frac{1}{j-jw-1} \right]$$

$$= e^{jwt} \left[j\frac{1}{w} - j\frac{e^{j10w}}{w} + \frac{e^{j-jw-1}}{-1+j-jw} - \frac{1}{-1+j-jw} \right]$$

$$= e^{jwt} \left[j\frac{1}{w} - j\frac{e^{j10w}}{w} + \frac{-e^{j-jw-1} - e^{j-jw-j\frac{\pi}{2}-1}(1-w) + 1 + e^{-j\frac{\pi}{2}}(1-w)}{1+(1-w)^2} \right]$$

2) $w = 10\pi$.

$$y(t) = e^{j10\pi t} \left[j\frac{1}{10\pi} - j\frac{1}{10\pi} + \frac{e^{j-j10\pi-1}}{-1+j-j10\pi} - \frac{1}{-1+j-j10\pi} \right]$$

$$= e^{j10\pi t} \left[\frac{1-e^{j-j10\pi-1}}{1+j(10\pi-1)} \right]$$

$y(t)_{w=10\pi}$ is periodic with fundenetal period to be $\frac{2\pi}{10\pi} = \frac{1}{5}$, Q3. Hint 2:

$$y(t) = \left(\frac{\cos wt + j\sin wt + \cos wt - j\sin wt}{5}\right)^2$$

$$= \left(\frac{2\cos wt}{5}\right)^2$$

$$= \frac{4}{25}\cos^2 wt$$

$$= \frac{4}{25}\frac{1}{2}(\cos 2wt + 1)$$

$y(t)$ is periodic and the period is $\frac{2\pi}{2\times 10\pi} = \frac{1}{10}$.

Question 4: [10%, Work-out question, Learning Objectives 1, 4, 5, and 6]
Consider the following DT signal.

$$x[n] = e^{j\frac{7\pi(n-\sqrt{2})}{8}} + \cos(\frac{\pi n - \sqrt{3}}{4}) + \sin(0.5\pi(n + \ln(2))). \tag{13}$$

Is $x[n]$ periodic? If so, find the fundamental period of $x[n]$. If not, carefully explained why $x[n]$ is not periodic.

This is NOT a yes/no question. Please carefully explain your answer.

Q4. $x_1[n] = e^{j\frac{7\pi(n-\sqrt{2})}{8}}$

Assume period to be N_1.

$$\frac{7\pi N_1}{8} = 2\pi \cdot k \quad k \in \mathbb{Z}.$$

$N_1 = \frac{16}{7}k$. let $k=7$. we get the fundemental period: $\underline{N_1 = 16}$.

$x_2[n] = \cos\left(\frac{\pi n - \sqrt{3}}{4}\right)$.

Assume period to be N_2.

$$\frac{\pi}{4}N_2 = 2\pi \cdot k \quad k \in \mathbb{Z}.$$

$N_2 = 8k$. Let $k=1$, the fundemental period $\underline{N_2 = 8}$.

$\sin(0.5\pi(n+\ln 12)))$

Assume period to be N_3.

$$\frac{1}{2}\pi N_3 = 2\pi k \quad k \in \mathbb{Z}$$

$N_3 = 4k$. let $k=1$. $\underline{N_3 = 4}$.

$\mathrm{LCM}(16, 8, 4) = 16$.

It is periodic and the fundemental period is $\underline{16}$.

Question 5: [12%, Work-out question, Learning Objective 1]

Consider the following system that takes a continuous-time signal $x(t)$ as input and outputs a discrete-time signal $y[n]$:

$$y[n] = n^2 \cdot \sin(0.5n) \cdot x(5\pi n - 1.5) \tag{14}$$

1. [12%] Is the above system is linear or not? Carefully explain the steps how you prove that the system is linear or not.

Q5.

Configuration 1:

$$\alpha_1 x_1[n] \rightarrow \boxed{\text{System}} \rightarrow \alpha_1 y_1[n] = \alpha_1 n^2 \sin(0.5n)\, x_1(5\pi n - 1.5)$$

$$\alpha_2 x_2[n] \rightarrow \boxed{\text{System}} \rightarrow \alpha_2 y_2[n] = \alpha_2 n^2 \sin(0.5n)\, x_2(5\pi n - 1.5)$$

$\oplus \rightarrow$

$$y[n] = \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

$$= n^2 \sin(0.5n)\left[\alpha_1 x_1(5\pi n - 1.5) + \alpha_2 x_2(5\pi n - 1.5)\right].$$

Configuration 2:

$\alpha_1 x_1[n]$

$\alpha_2 x_2[n]$

$$\oplus \rightarrow x_3[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n] \rightarrow \boxed{\text{System}} \rightarrow$$

$$y_3[n] = n^2 \sin(0.5n)\left[\alpha_1 x_1[5\pi n - 1.5] + \alpha_2 x_2(5\pi n - 1.5]\right]$$

$$y[n] = y_3[n]$$

It is linear.

Question 6: [20%, Multiple Choices, Learning Objectives 1 and 6]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider two continuous-time signals:

$$x_1(t) = \int_{s=t-5}^{t+5} \left(\cos(\frac{2\pi}{10}s) + \cos(\frac{2}{10}s)\right) ds \quad (15)$$

$$x_2(t) = \frac{e^{j|t|} + e^{-j|t|}}{2} \cdot (\cos(2t))^2 \quad (16)$$

and two discrete-time signals:

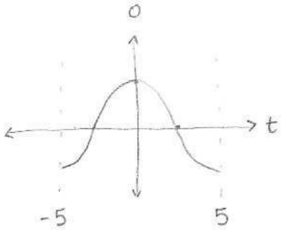
$$x_3[n] = \sum_{k=-4}^{3} (-1)^k \cdot \delta[n-2+4k] \quad (17)$$

$$x_4[n] = \sum_{k=-\infty}^{\infty} 3^{-|k|} e^{j\pi(n-k)}. \quad (18)$$

1. [10%] For $x_1(t)$ to $x_4[n]$, determine whether it is periodic or not, *respectively.* If it is periodic, write down the fundamental period. Please state explicitly which signal is periodic and which is not.
2. [10%] For $x_1(t)$ to $x_4[n]$, determine whether it is even or odd or neither of them, respectively. Please state explicitly which signal is even, which is odd, and which is neither.

1. $x_1(t) \longrightarrow \boxed{\text{periodic, } T = 10\pi}$

$$x_1(t) = \underbrace{\int_{t-5}^{t+5} \cos\left(\frac{2\pi}{10}s\right) ds}_{\text{period} = 10} + \int_{t-5}^{t+5} \cos\left(\frac{2}{10}s\right) ds$$



→ integrating cosine over 1 period = 0

$$x_1(t) = \frac{10}{2}\left[\sin\left(\frac{2}{10}s\right)\right]_{s=t-5}^{s=t+5}$$

$$x_1(t) = 5\sin\left(\frac{1}{5}(t+5)\right) - 5\sin\left(\frac{1}{5}(t-5)\right)$$

both terms have $T = \frac{2\pi}{1/5} = 10\pi$

$x_2(t) \rightarrow$ periodic, $T = 2\pi$

$x_3[n] \rightarrow$ aperiodic

$x_4[n] \rightarrow$ periodic, $N = 2$

$$x_2(t) = \cos(|t|)\cos(2t)^2$$

$$= \underbrace{\cos(t)}_{T=2\pi}\underbrace{\cos(2t)^2}_{T=\pi}$$

$$\mathrm{LCM}(2\pi, \pi) = 2\pi$$

$$x_4[n] = e^{j\pi n} \underbrace{\sum_{k=-\infty}^{\infty} 3^{-|k|} e^{-j\pi k}}_{\text{constant for each term in summation}}$$

$$x_4[n] = \ldots \underset{(k=-1)}{-\frac{1}{3} e^{j\pi n}} + \underset{(k=0)}{e^{j\pi n}} - \underset{(k=1)}{\frac{1}{3} e^{j\pi n}} + \underset{(k=2)}{\frac{1}{9} e^{j\pi n}} \ldots$$

$$N = \frac{2\pi}{\pi} = 2$$

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2. $x_1(t) \rightarrow$ even

$$x_1(-t) = \int_{-t-5}^{-t+5} \cos\left(\frac{2}{10} s\right) ds$$

$$x_1(-t) = \int_{t-5}^{t+5} \cos\left(\frac{2}{10} s\right) ds$$

$$x_1(-t) = x_1(t)$$

$x_2(t) \rightarrow$ even

$x_3[n] \rightarrow$ neither

$x_4[n] \rightarrow$ even

$$e^{j\pi n} = \underset{\text{even}}{\cos(\pi n)} + j\underset{=0}{\sin(\pi n)}$$