Midterm #2 of ECE301, Section 3 (CRN 17102-004)
6:30–7:30pm, Thursday, October 11, 2018, PHYS112.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: SOLUTION

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [22%, Work-out question, Learning Objectives 1, 2, and 3]

1. [4%] What is the definition of *impulse response*?
2. [12%] Consider a DT-LTI system with the input/output relationship being

$$y[n] = \sum_{k=-\infty}^{n-2} x[k]e^{n-k-2} \tag{1}$$

   Find the impulse response $h[n]$.

3. [6%] Is the above system causal? Please carefully write down your reasons. A yes/no answer without reasons will not receive any point.

   Hint: If you do not know the answer to the previous subquestion, you can assume $h[n] = 3 - U[n] - 2U[n-2]$. You will receive full points if your answer is correct. However, please indicate carefully if you are solving the alternative question.

1. The impulse response of a system is the output when the input is the unit impulse function.

2. $x[n] = \partial[n]$

$$h[n] = \sum_{k=-\infty}^{n-2} \partial[k] e^{n-k-2} = e^{n-0-2} \text{ for } n-2 \geq 0$$

$$\boxed{h[n] = \begin{cases} e^{n-2}, & n \geq 2 \\ 0, & \text{else} \end{cases}}$$

3. Yes because $h[n] = 0$ for all $n < 0$.

Question 2: [15%, Work-out question, Learning Objectives 1, 2, and 3]

Consider the following ~~three~~ two signals:

$$x_1(t) = e^{-|t|} \tag{2}$$

$$x_2(t) = e^{j3t} \tag{3}$$

1. [15%] Compute the convolution $y(t) = x_1(t) * x_2(t)$.

$$y(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$x_1(\tau) = \begin{cases} e^{\tau}, & \tau < 0 \\ e^{-\tau}, & \tau > 0 \end{cases} \qquad x_2(t-\tau) = e^{j3(t-\tau)}$$

$$y(t) = \int_{-\infty}^{0} e^{\tau} e^{j3t - j3\tau}\, d\tau + \int_{0}^{\infty} e^{-\tau} e^{j3t - j3\tau}\, d\tau$$

$$= e^{j3t} \int_{-\infty}^{0} e^{\tau(1-3j)}\, d\tau + e^{j3t} \int_{0}^{\infty} e^{\tau(-1-3j)}\, d\tau$$

$$= \frac{e^{j3t}}{1-3j} \left[e^{\tau(1-3j)} \right]_{\tau=-\infty}^{0} + \frac{e^{j3t}}{-1-3j} \left[e^{\tau(-1-3j)} \right]_{\tau=0}^{\infty}$$

$$= \frac{e^{j3t}}{1-3j} [1 - 0] + \frac{e^{j3t}}{-1-3j} [0 - 1]$$

$$\boxed{y(t) = \frac{e^{j3t}}{1-3j} + \frac{e^{j3t}}{1+3j}}$$

Question 3: [15%, Work-out question, Learning Objectives 2, 3, and 4]
Consider a CT-LTI system with impulse response

$$h(t) = \begin{cases} e^{-2t} & \text{if } 0 < t \\ e^{t} & \text{if } t \leq 0 \end{cases} \tag{4}$$

Find the output $y(t)$ when then input is $x(t) = e^{j3t} + e^{-j0.5\pi t}$.

Hint: There is no need to greatly simplify your answers. Some expressions of the form of $\frac{3}{3+4j}e^{j\sqrt{2}t}$ would suffice.

$x_1(t) = e^{j3t}$ $\quad x_2(t) = e^{-j0.5\pi t}$

$y(t) = x(t) * h(t)$

$y(t) = (x_1(t) + x_2(t)) * h(t)$

$y(t) = x_1(t) * h(t) + x_2(t) * h(t)$

$$= \int_{-\infty}^{0} e^{j3(t-\tau)} e^{\tau} d\tau \int_{0}^{\infty} e^{j3(t-\tau)} e^{-2\tau} d\tau + \int_{-\infty}^{0} e^{-j0.5\pi(t-\tau)} e^{\tau} d\tau + \int e^{-j0.5\pi(t-\tau)} e^{-2\tau} d\tau$$

$$= e^{j3t}\int_{-\infty}^{0} e^{\tau(1-3j)} d\tau + e^{j3t}\int_{0}^{\infty} e^{\tau(-3j-2)} d\tau + e^{-j\frac{\pi}{2}t}\int_{-\infty}^{0} e^{\tau(j\frac{\pi}{2}+1)} + e^{-j\frac{\pi}{2}t}\int_{0}^{\infty} e^{\tau(j\frac{\pi}{2}-2)}$$

$$= \frac{e^{j3t}}{1-3j}\left[e^{\tau(1-3j)}\right]_{-\infty}^{0} - \frac{e^{j3t}}{3j+2}\left[e^{\tau(-3j-2)}\right]_{0}^{\infty} + \frac{e^{-j\frac{\pi}{2}t}}{j\frac{\pi}{2}+1}\left[e^{\tau(j\frac{\pi}{2}+1)}\right]_{-\infty}^{0} + \frac{e^{-j\frac{\pi}{2}t}}{j\frac{\pi}{2}-2}\left[e^{\tau(j\frac{\pi}{2}-2)}\right]_{0}^{\infty}$$

$$= \boxed{\frac{e^{j3t}}{1-3j} + \frac{e^{j3t}}{3j+2} + \frac{e^{-j\frac{\pi}{2}t}}{j\frac{\pi}{2}+1} - \frac{e^{-j\frac{\pi}{2}t}}{j\frac{\pi}{2}-2}}$$

Question 4: [20%, Work-out question, Learning Objectives 4 and 5]

1. [12%] Consider a periodic CT signal

$$x(t) = \cos(3.5\pi t + 1) + \sin(2.5\pi t). \tag{5}$$

   Find the Fourier series representation of $x(t)$.

   Hint: The following formula may be useful:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{6}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}. \tag{7}$$

2. [8%] Continue from the previous question. Consider another CT signal $y(t) = \frac{d}{dt}x(t)$. Find the Fourier series representation of $y(t)$.

   Hint: If you do not know the answer to the previous question, you can assume $y(t) = \sin(0.5\pi t)$. You will receive 4 points if your answer is correct.

1. $x(t) = \dfrac{e^{j(3.5\pi t + 1)} + e^{-j(3.5\pi t + 1)}}{2} + \dfrac{e^{j2.5\pi t} - e^{-j2.5\pi t}}{2j}$

$T = LCM\left(\frac{2\pi}{3.5\pi}, \frac{2\pi}{2.5\pi}\right) = LCM\left(\frac{4}{7}, \frac{4}{5}\right) = 4$

$x(t) = \frac{e^{j(1)}}{2} e^{j\frac{\pi}{2}(7)} + \frac{e^{-j(1)}}{2} e^{-j\frac{\pi}{2}(7)} + \frac{1}{2j} e^{j\frac{\pi}{2}(5)} - \frac{1}{2j} e^{-j\frac{\pi}{2}(5)}$

$a_7 = \frac{e^{j}}{2}$

$a_{-7} = \frac{e^{-j}}{2}$

$a_5 = \frac{1}{2j}$

$a_{-5} = -\frac{1}{2j}$

$a_k = 0$ for all other k

Question 5: [8%, Work-out question, Learning Objective 4] Consider a DT signal

$$x[n] = \begin{cases} 2\pi & \text{if } 0 \leq n \leq 2 \\ 0 & \text{if } 3 \leq n \leq 49\ . \\ x[n] \text{ is periodic with period 50} & \end{cases} \tag{8}$$

Find the DTFS representation of $x[n]$.

$$a_k = \frac{1}{N} \sum^{N} x[n] e^{-jk2\pi \frac{n}{N}}$$

$$a_k = \frac{2\pi}{50} \sum_{n=0}^{2} e^{-j\frac{k2\pi n}{50}}$$

$$a_k = \frac{\pi}{25} \sum_{n=0}^{2} e^{-j\frac{k\pi n}{25}}$$

$$\boxed{a_k = \frac{\pi}{25}\left(1 + e^{-j\frac{k\pi}{25}} + e^{-jk\frac{2\pi}{25}}\right)}$$

Question 6: [20%, Multiple Choices, Learning Objective 1]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \int_{-\infty}^{t-2} e^{-|x_1(s)|-|s|} ds \tag{9}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = \begin{cases} \int_{s=-\infty}^{x_2[n]} e^s ds & \text{if } n \leq 0 \\ \int_{s=x_2[n]}^{\infty} e^{-s} ds & \text{if } n > 0 \end{cases} \tag{10}$$

Answer the following questions

1. [4%, Learning Objective 1] Is System 1 memoryless? Is System 2 memoryless?
2. [4%, Learning Objective 1] Is System 1 causal? Is System 2 causal?
3. [4%, Learning Objective 1] Is System 1 stable? Is System 2 stable?
4. [4%, Learning Objective 1] Is System 1 linear? Is System 2 linear?
5. [4%, Learning Objective 1] Is System 1 time-invariant? Is System 2 time-invariant?

	System 1	System 2
1.	no	yes
2.	yes	yes
3.	yes	yes
4.	no	no
5.	no	no

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(j\omega) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau) y(t - \tau) d\tau$	$T a_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\, dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t - 1) - 1/2.$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$