Solution

Midterm #3 of ECE301, Section 3 (CRN 17102-004)
8–9pm, Wednesday, September 12, 2018, FRNY G140.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: SOLUTION

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Date:

Question 1: [18%, Work-out question, Learning Objectives 1 and 2]

1. [8%] Consider a CT signal

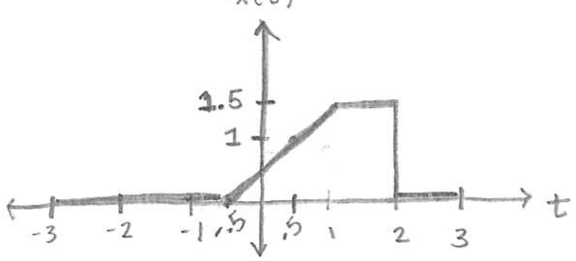
$$x(t) = (t + 0.5)\mathcal{U}(t + 0.5) - (t - 1)\mathcal{U}(t - 1) - 1.5\mathcal{U}(t - 2) \tag{1}$$

   Plot $x(t)$ for the range of $-3 < t < 3$.

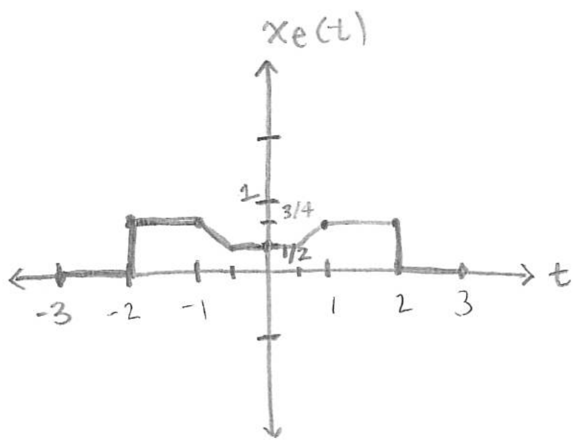
2. [10%] Plot the even part of $x(t)$ for the range of $-3 < t < 3$.

   Hint: If you do not know how to solve this question, write down the formulas of the odd part of $x(t)$. You will receive 4 points if your answer is correct.

1.



2. $x_e(t) = \dfrac{x(t) + x(-t)}{2}$ $\qquad x_o(t) = \dfrac{x(t) - x(-t)}{2}$



t	$x(t)$	$x(-t)$	$x_e(t)$
-3	0	0	0
-2	0	1.5	0.75
-1	0	1.5	0.75
0	0.5	0.5	0.5
-0.5	0	1	0.5

Question 2: [16%, Work-out question, Learning Objective 3]
Define two continuous-time signals:

$$x(t) = \begin{cases} e^{2jt} & \text{if } 2 \leq t \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and

$$y(t) = \begin{cases} e^{\pi t} & \text{if } t \leq 2 \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

respectively. Compute the expression of the following sum

$$z(t) = \int_{s=-\infty}^{\infty} x(s)y(0.5(t-s))ds. \tag{4}$$

Hint: You can leave your answer to be of the form like $\frac{e^{3j}-2^{-12}}{3+10j}$. There is no need to further simplify the expression.

$$x(s) = \begin{cases} e^{j2s} & , 2 \leq s \\ 0 & , \text{else} \end{cases} \qquad y(0.5(t-s)) = \begin{cases} e^{\frac{\pi}{2}(t-s)} & , \frac{1}{2}t - \frac{1}{2}s \leq 2 \\ 0 & , \text{else} \end{cases}$$

$$t - s \leq 4$$

$$t - 4 \leq s$$

$\boxed{\text{Case 1:}}$ $t - 4 \leq 2$

$t \leq 6$

$$\int_2^{\infty} e^{j2s} e^{\frac{\pi}{2}(t-s)} ds$$

$$= e^{\frac{\pi}{2}t} \int_2^{\infty} e^{(j2 - \frac{\pi}{2})s} ds = \frac{e^{\frac{\pi}{2}t}}{j2 - \frac{\pi}{2}} \left[e^{(j2-\frac{\pi}{2})s} \right]_{s=2}^{s=\infty}$$

$$= \frac{e^{\frac{\pi}{2}t}}{2j - \frac{\pi}{2}} \left[0 - e^{4j-\pi} \right] = \frac{e^{\frac{\pi}{2}t} e^{4j-\pi}}{\frac{\pi}{2} - 2j}$$

$\boxed{\text{Case 2:}}$ $t-4>2$

$t>6$

$$\int_{t-4}^{\infty} e^{j2s} e^{\frac{\pi}{2}(t-s)}\, ds$$

Note:

$$\lim_{s\to\infty} e^{j2s} e^{-\frac{\pi}{2}s}$$

$$= \lim_{s\to\infty} \cancelto{0}{e^{-\frac{\pi}{2}s}} (\cos(2s)+j\sin(2s))$$

$$=0$$

$$= \frac{e^{\frac{\pi}{2}t}}{j2-\frac{\pi}{2}} \left[e^{j2s} e^{-\frac{\pi}{2}s} \right]_{s=t-4}^{s=\infty}$$

$$= \frac{-e^{\frac{\pi}{2}t} e^{j2(t-4)} e^{-\frac{\pi}{2}(t-4)}}{j2-\frac{\pi}{2}} = \frac{e^{j2t} e^{-j8+2\pi}}{\frac{\pi}{2}-2j}$$

$$z(t) = \begin{cases} \dfrac{e^{\frac{\pi}{2}t} e^{4j-\pi}}{\frac{\pi}{2}-2j}, & t \le 6 \\[2ex] \dfrac{e^{j2t} e^{-j8+2\pi}}{\frac{\pi}{2}-2j}, & t > 6 \end{cases}$$

Question 3: [20%, Work-out question, Learning Objectives 1, 4, and 5] Given a discrete-time signal

$$x[n] = \begin{cases} 2^n e^{j\pi(n+10)} & \text{if } n \leq 20 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Based on $x[n]$, we construct another function

$$y(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}. \tag{6}$$

1. [10%] Compute the average power $x[n]$ for the duration of $n = -100$ to 100.
2. [10%] Compute the value of $y(1+j)$.

Hint 1: $\cos(\pi/\overset{4}{\cancel{2}}) = \sin(\pi/\overset{4}{\cancel{2}}) = \sqrt{2}/2$
Hint 2: You may need to use the following two formulas: If $|r| \neq 1$, then we have

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}. \tag{7}$$

If $|r| < 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}. \tag{8}$$

If $|r| \geq 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} \text{ does not exist.} \tag{9}$$

Hint 3: You can leave your answer to be of the form like $\frac{e^{3j}-2^{-12}}{3+10j}$. There is no need to further simplify the expression.

1. $\frac{1}{201}\sum_{n=-100}^{20}\left|2^n e^{j\pi(n+10)}\right|^2$

Note: The magnitude of a complex exponential is always 1, and 2^n cannot be neg.

$= \frac{1}{201}\sum_{n=-100}^{20} 2^{2n} = \frac{1}{201}\sum_{n=-100}^{20} 4^n \longrightarrow$ Let $\ell = n+101$ $(n = \ell - 101)$

$= \frac{1}{201}\sum_{\ell=1}^{121} 4^{\ell-1-100} = \frac{4^{-100}}{201}\sum_{\ell=1}^{121} 4^{\ell-1} \longrightarrow$ *use formula $(K=121)$

$= \frac{4^{-100}}{201}\left(\frac{1-4^{121}}{-3}\right) = \frac{4^{-100}-4^{21}}{-603} = \boxed{\frac{4^{21}-4^{-100}}{603}}$

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2. $y(1+j) = y\left(\frac{2}{\sqrt{2}}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)\right) = y\left(\sqrt{2}\, e^{j\frac{\pi}{4}}\right)$

$= \sum_{n=-\infty}^{20} 2^n e^{j\pi(n+10)}\left(\sqrt{2}e^{j\frac{\pi}{4}}\right)^{-n} = e^{j\pi 10}\sum_{n=-\infty}^{20} 2^{n/2} e^{j(\pi-\frac{\pi}{4})n}$

$= (1)\sum_{n=-\infty}^{20}\left(e^{\frac{\ln 2}{2}} e^{j\frac{3\pi}{4}}\right)^n \longrightarrow$ Let $\ell = -n+21$ $(n = 21-\ell)$

$= \sum_{\ell=1}^{\infty}\left(e^{\left(\frac{\ln 2}{2}+j\frac{3\pi}{4}\right)}\right)^{21-\ell} = \sum_{\ell=1}^{\infty}\left(e^{-\left(\frac{\ln 2}{2}+j\frac{3\pi}{4}\right)}\right)^{\ell-1-20}$

$= \frac{e^{-20\left(-\frac{\ln 2}{2}\right)} e^{-20\left(-j\frac{3\pi}{4}\right)}}{1 - e^{-\frac{\ln 2}{2}} e^{-j\frac{3\pi}{4}}} = \frac{e^{10\ln 2}\, e^{15\pi}}{1-\left(2^{-\frac{1}{2}}\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)\right)} = \frac{2^{10}(-1)}{1-\left(-\frac{1}{2}-j\frac{1}{2}\right)}$

$= \frac{-2^{10}}{\frac{3}{2}+\frac{j}{2}} = \boxed{\frac{-2^{11}}{3+j}}$

Question 4: [10%, Work-out question, Learning Objectives 1, 4, 5, and 6]
Consider the following DT signal.

$$x[n] = \cos(\frac{8\pi(n-3)}{5}) + \sin(\frac{12\pi n}{5}) + \cos(\pi(n+4)). \tag{10}$$

Is $x[n]$ periodic? If so, find the fundamental period of $x[n]$. If not, carefully explained why $x[n]$ is not periodic.

This is NOT a yes/no question. Please carefully explain your answer.

x[n] is periodic

$\frac{8\pi}{5} N = 2\pi$ $\qquad$ $\frac{12\pi}{5} N = 2\pi$ $\qquad$ $\pi N = 2\pi$

$N = \frac{5}{4}$ $\qquad$ $N = \frac{5}{6}$ $\qquad$ $N = 2$

(DT, so $N = 5$) $\qquad$ (DT, so $N = 5$)

$\text{LCM}(5, 5, 2) = 10$

so the fundamental period of x[n] is 10

Question 5: [16%, Work-out question, Learning Objective 1]

Consider the following system that takes a discrete-time signal $x[n]$ as input and outputs a discrete-time signal $y[n]$:

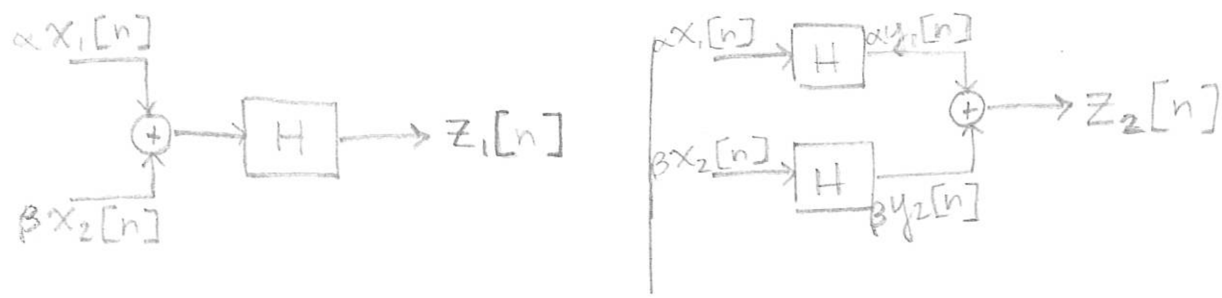
$$y[n] = \sum_{k=-\infty}^{100} \mathcal{U}[k]x[2n^2 - k] \tag{11}$$

1. [16%] Is the above system is linear or not? Carefully explain the steps how you prove that the system is linear or is not linear.

Call the system H and define

$y_1[n] = H(x_1[n])$ and $y_2[n] = H(x_2[n])$

if linear, then these two diagrams are equivalent:



or $z_1[n] = z_2[n]$

$$H(\alpha x_1[n] + \beta x_2[n]) = \alpha y_1[n] + \beta y_2[n]$$

$$\sum_{k=-\infty}^{\infty} u[k](\alpha x_1 + \beta x_2)[2n^2 - k] = \sum_{k=-\infty}^{\infty} u[k]\alpha x_1[2n^2 - k] + \sum_{k=-\infty}^{\infty} u[k]\beta x_2[2n^2 - k]$$

$$\sum_{k=-\infty}^{\infty} u[k]\alpha x_1[2n^2 - k] + u[k]\beta x_2[2n^2 - k] = \sum_{k=-\infty}^{\infty} u[k]\alpha x_1[2n^2 - k] + u[k]\beta x_2[2n^2 - k]$$

This equality holds, so the system is linear.

Question 6: [20%, Multiple Choices, Learning Objectives 1 and 6]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider two continuous-time signals:

$$x_1(t) = \sum_{k=-1}^{3} 2^{-k} \sin((\pi + k\pi)t) \quad (12)$$

$$x_2(t) = \frac{e^{|t|} - e^{-|t|}}{2} \cdot (\sin(1.5t))^2 \quad (13)$$

and two discrete-time signals:

$$x_3[n] = \sin(0.75\pi n^2) + \cos(\frac{\pi n}{2}) \quad (14)$$

$$x_4[n] = \sum_{k=-\infty}^{n} 2^k \sin(0.2k) + \sum_{k=-\infty}^{-n} 2^k \sin(0.2k). \quad (15)$$

1. [10%] For $x_1(t)$ to $x_4[n]$, determine whether it is periodic or not, *respectively.* If it is periodic, write down the fundamental period. Please state explicitly which signal is periodic and which is not.

2. [10%] For $x_1(t)$ to $x_4[n]$, determine whether it is even or odd or neither of them, respectively. Please state explicitly which signal is even, which is odd, and which is neither.

1. periodic $\longrightarrow T = LCM\left(\frac{2\pi}{\pi}, \frac{2\pi}{2\pi}, \frac{2\pi}{3\pi}, \frac{2\pi}{4\pi}\right) = LCM\left(2, 1, \frac{2}{3}, \frac{1}{2}\right) \rightarrow \boxed{T=2}$

aperiodic

periodic $\rightarrow T = LCM(4, 2) = \boxed{4}$

aperiodic (can't find N such that $x_4[n] = x_4[n+N]$ for all n)

2. odd (sum of odd signals)

even (product of even signals)

even (sum of even signals)

even ($x[n] = x[-n]$)