Midterm #3 of ECE301, Section 3 (CRN 17101-003)
8–9pm, Thursday, November 16, 2017, ME1061 170.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [22%, Work-out question, Learning Objectives 4, 5] Consider the following discrete-time periodic signal $x[n]$:

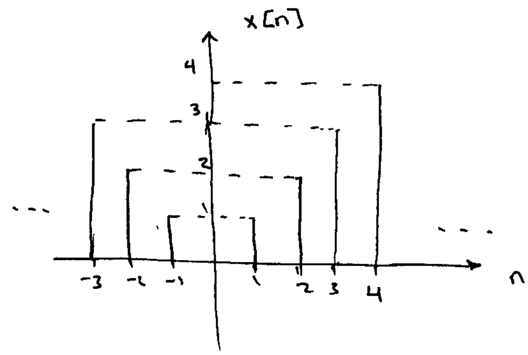
$$x[n] = \begin{cases} |n| & \text{if } -3 \leq n \leq 4 \\ \text{periodic with period 8} & \end{cases}. \tag{1}$$

Let a_k denote the Discrete-Time Fourier Series coefficients of $x[n]$.

1. [6%] Find the value of a_0.
2. [7%] Find the value of $\sum_{k=0}^{7} j^k a_k$. Hint: The Fourier series synthesis formula will be useful.

Also define $y[n] = (1 - e^{j\frac{3\pi}{4}n})$ and let $z[n] = y[n] \cdot x[n]$.

3. [9%] Denote the Discrete-Time Fourier Series coefficients of $z[n]$ by c_k. Find the expression of c_2 in terms of a_k. Namely, you can assume a_k is known completely and write down your expression of c_7 in terms of a_k.



1) $a_k = \frac{1}{N} \sum_{n=-3}^{4} x[n]\, e^{-jk\frac{2\pi}{N}n}$

$a_0 = \frac{1}{8} \sum_{n=-3}^{4} |n| \cancel{e}$

$= \frac{1}{8}(3+2+1+1+2+3+4) = \frac{16}{8} = 2$

2) Synthesis formula: $x[n] = \sum_{n=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$

* remark: $e^{j\frac{\pi}{2}} = j$

$\sum_{k=0}^{7} j^k a_k$, comparing the formulas:

$j^k = e^{jk\frac{2\pi}{N}n} \Rightarrow n = 2$

$x[\boxed{2}] = \sum_{k=0}^{7} a_k e^{jk(\frac{2\pi}{8})2} = \sum_{k=0}^{7} j^k a_k = 2$

3) $y[n] = 1 - e^{j\frac{3\pi}{4}n}$

$$Z[n] = y[n] \cdot x[n]$$

$$= \left(1 - e^{j\frac{3\pi}{4}n}\right) x[n]$$

$$= x[n] - e^{j\frac{3\pi n}{4}} x[n]$$

$$= x[n] - e^{j3\left(\frac{2\pi}{8}\right)n} x[n]$$

using frequency shift and linearity:

$$c_k = a_k - a_{k-3}$$

$$\boxed{c_2 = a_2 - a_{-1}}$$

Question 2: [18%, Work-out question, Learning Objective 4] Consider the following differential CT signal:

$$y(t+1.5) = \frac{d}{dt}x(t) + 0.5\frac{d^2}{dt^2}y(t). \tag{2}$$

1. [6%] Find the *frequency response* $H(j\omega)$ of the above system.
2. [3%] Suppose the input signal is $x(t) = \cos(2t)$. Find the Fourier series representation of $x(t)$.
3. [9%] Suppose the input signal is $x(t) = \cos(2t)$. Find the Fourier series representation of the output $y(t)$.

   Hint: if you do not know the $H(j\omega)$ in the previous question, you can assume $H(j\omega) = \frac{e^{-j\omega/2}\omega^4}{1+3\omega^2}$. You will receive full credit if your answer is correct.

1) $y(t+\frac{3}{2}) = \frac{d}{dt}x(t) + \frac{1}{2}\frac{d^2}{dt^2}y(t)$

$e^{j\omega\frac{3}{2}}Y(j\omega) = j\omega X(j\omega) + \frac{1}{2}j^2\omega^2 Y(j\omega)$

$Y(j\omega)\left(e^{j\omega\frac{3}{2}} + \frac{\omega^2}{2}\right) = j\omega X(j\omega)$

$$\boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega}{e^{j\omega\frac{3}{2}} + \frac{\omega^2}{2}}}$$

2) $x(t) = \cos(2t)$

$= \frac{e^{j2t}}{2} + \frac{e^{-j2t}}{2}$ $\qquad \omega = 2$

$$\boxed{a_1 = \frac{1}{2}, \quad a_{-1} = \frac{1}{2}, \quad a_k = 0 \text{ otherwise}}$$

3) $y(t) = H(j\omega)\cdot x(t)$

$b_k = H(j\omega)\cdot a_k$

$b_1 = H(j2)\cdot a_1$ $\qquad b_{-1} = H(j(-2))\, a_{-1}$

$H(j2) = \frac{2j}{e^{j3}+2}$ $\qquad H(-2j) = \frac{-2j}{e^{-j3}+2}$ $\qquad b_k = 0$, for all other k

$\boxed{b_1 = \frac{j}{e^{j3}+2}}$, $\qquad \boxed{b_{-1} = \frac{-j}{e^{-j3}+2}}$

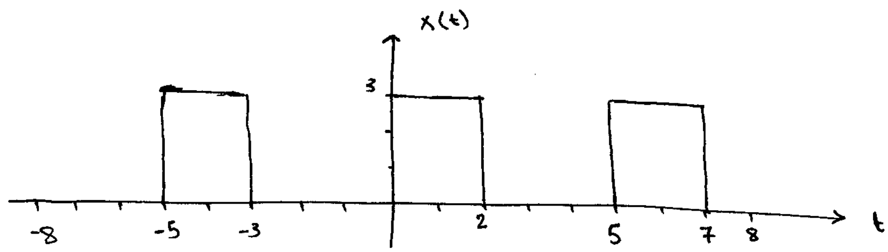
Question 3: [15%, Work-out question, Learning Objectives 1, 2, 3, 4, 5, 6] Consider a continuous time signal $x(t)$:

$$x(t) \begin{cases} 3 & \text{if } 0 \le t < 2 \\ 0 & \text{if } 2 \le t < 5\,. \\ \text{periodic with period 5} \end{cases} \tag{3}$$

1. [3%] Plot $x(t)$ for the range of $-8 < t < 8$.

2. [12%] Find the Fourier transform $X(j\omega)$. Hint: If you do not know how to solve this question, you can find the CTFS of $x(t)$ instead. You will receive 8 points if your answer is correct.

1)



2) $a_k = \frac{1}{T}\int x(t)\, e^{-jk\frac{2\pi t}{T}}\, dt$

$= \frac{1}{5}\int_0^2 3\, e^{-jk\frac{2\pi}{5}t}\, dt$

$= \frac{3}{5}\cdot\frac{1}{-jk\frac{2\pi}{5}}\, e^{-jk\frac{2\pi}{5}t}\Big]_0^2$

$= \frac{-3}{+jk2\pi}\left(e^{-jk\frac{4\pi}{5}} - 1\right)$

since $x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\frac{2\pi}{5}t}$

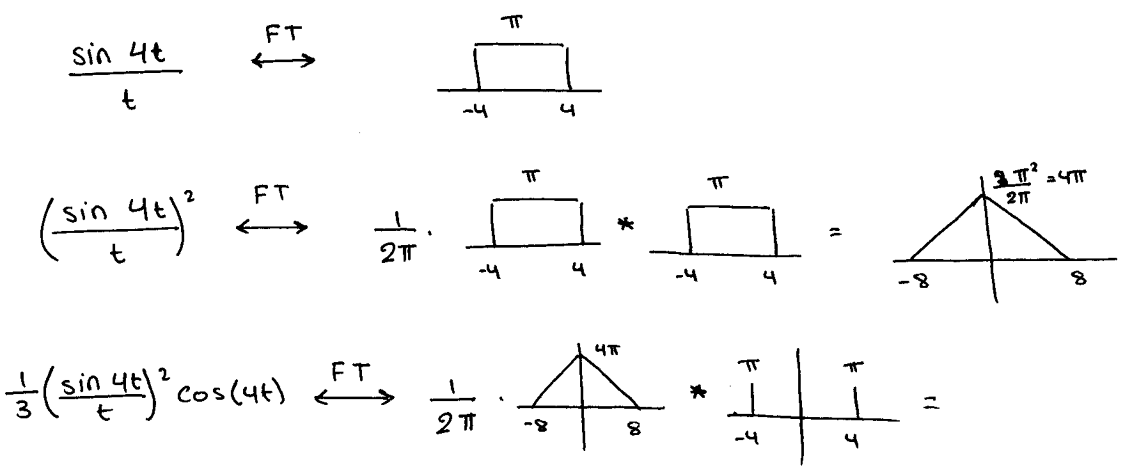
then $X(j\omega) = \sum_{k=-\infty}^{\infty} a_k \cdot 2\pi \cdot \delta\left(\omega - \frac{k2\pi}{5}\right)$

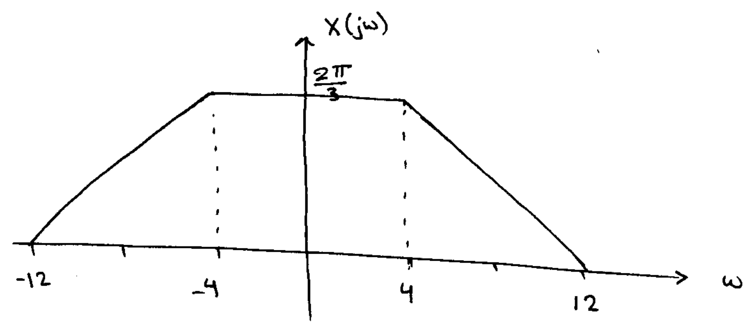
$$\boxed{X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{3}{jk}\left(1 - e^{-jk\frac{4\pi}{5}}\right)\delta\left(\omega - \frac{k2\pi}{5}\right)}$$

Question 4: [13%, Work-out question, Learning Objectives 4, 5] Consider the following signal $x(t)$

$$x(t) = \frac{(\sin(4t))^2}{3t^2}\cos(4t) \tag{4}$$

Find the corresponding Fourier transform $X(j\omega)$ and plot it for the range of $-12 < \omega < 12$.





Question 5: [12%, Work-out question, Learning Objectives 4, 5, and 6] Consider a discrete time signal

$$x[n] = 2^{-|n|}e^{j3n} \tag{5}$$

Find the corresponding Fourier transform $X(e^{j\omega})$.

Hint: You may need the following formula: If $|r| \neq 1$, then

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}. \tag{6}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} 2^{-|n|} e^{j3n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} 2^{n} e^{j3n} e^{-j\omega n} + \sum_{n=0}^{\infty} 2^{-n} e^{j3n} e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2e^{j(3-\omega)}}\right)^{n} + \sum_{n=1}^{\infty} \left(\frac{e^{j(3-\omega)}}{2}\right)^{n-1}$$

$$= \frac{1}{2e^{j(3-\omega)}} \cdot \frac{1}{1-\frac{1}{2e^{j(3-\omega)}}} + \frac{1}{1-\frac{e^{j(3-\omega)}}{2}}$$

$$\boxed{X(e^{j\omega}) = \frac{1}{2e^{j(3-\omega)}-1} + \frac{2}{2-e^{j(3-\omega)}}}$$

Question 6: [20%, Learning Objectives 2, 3, 4, and 5] Consider the following AM transmission system. The input signal is $x(t) = \cos(5.5t)$. We first multiply $x(t)$ by $\cos(3t)$. That is,

$$y(t) = x(t) \cdot \cos(3t).$$

The transmitter then transmits signal $y(t)$ through the antenna.

At the receiver side, we first multiply $y(t)$ by $2\cos(3t)$. That is $z(t) = y(t) \cdot 2\cos(3t)$ and then pass $z(t)$ through a low pass filter with cutoff frequency $W = 1.5$ rad/sec. Denote the final output by $w(t) = z(t) * h_{\text{LPF}}(t)$.

Find the expression of $w(t)$.

Hint: If you do not know how to solve this question, you can assume $x(t) = \cos(0.001t)$ instead, and solve the following alternative question: Let $v(t)$ denote the output of the asynchronous AM demodulation. Plot $v(t)$ for the range $-5000\pi < t < 5000\pi$. If your answer is correct, you will receive 14 points.

Time - Domain analysis :

$$y(t) = \cos(5.5t) \cdot \cos(3t)$$

$$= \frac{1}{2}\cos(8.5t) + \frac{1}{2}\cos(2.5t)$$

$$z(t) = 2\cos(3t)\left(\frac{1}{2}\cos(8.5t) + \frac{1}{2}\cos(2.5t)\right)$$

$$= \frac{1}{2}\cos(11.5t) + \frac{1}{2}\cos(0.5t) + \cos(5.5t)$$

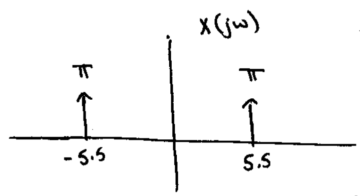
$$h_{LPF}(t) = \frac{\sin(1.5t)}{\pi t}$$

$$w(t) = z(t) * h_{LPF}$$

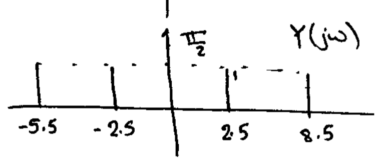
$$\boxed{w(t) = \frac{1}{2}\cos(0.5t)}$$

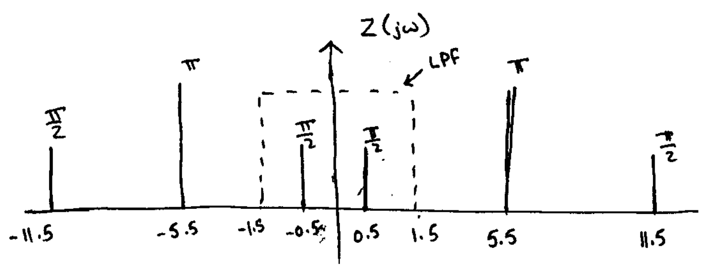
OR Frequency - Domain analysis :

$$X(j\omega) = \pi\left[\delta(\omega - 5.5) + \delta(\omega + 5.5)\right]$$



$$Y(j\omega) = X(j\omega) * \left(\pi\left[\delta(\omega - 3) + \delta(\omega + 3)\right]\right)$$





$$\therefore W(j\omega) = \frac{\pi}{2}\left[\delta(\omega - 0.5) + \delta(\omega + 0.5)\right]$$

$$\boxed{w(t) = \frac{1}{2}\cos(0.5t)}$$