Midterm #2 of ECE301, Section 3 (CRN 17101-003)
8–9pm, Wednesday, October 11, 2017, EE 170.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solutions

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [11%, Work-out question, Learning Objective 1] Consider a CT system with the input/output relationship being

$$y(t) = \begin{cases} \int_{t-1}^{t} e^{-x(s)} ds & \text{if } x(t) \geq 0 \\ 0 & \text{if } x(t) < 0 \end{cases} \quad (1)$$

Is this system time-invariant or not? Please carefully write down your reasons. A yes/no answer without reasons will not receive any point.

$$y_1(t) = y(t-t_0) = \begin{cases} \int_{t-t_0-1}^{t-t_0} e^{-x(s)} ds & , x(t-t_0) \geq 0 \\ 0 & , x(t-t_0) < 0 \end{cases}$$

$$x_2(t) = x(t-t_0)$$

$$y_2(t) = \begin{cases} \int_{t-1}^{t} e^{-x(s-t_0)} ds & , x(t-t_0) \geq 0 \\ 0 & , x(t-t_0) < 0 \end{cases}$$

let $z = s - t_0$

$$y_2(t) = \begin{cases} \int_{t-1-t_0}^{t-t_0} e^{-x(z)} dz & , x(t-t_0) \geq 0 \\ 0 & , x(t-t_0) < 0 \end{cases}$$

since $y_1(t) = y_2(t)$

$$\boxed{\text{System is Time-Invariant}}$$

Question 2: [15%, Work-out question, Learning Objectives 1, 2, and 3]
Consider the following CT system:

$$y(t) = \int_{s=-\infty}^{\infty} (U(s+1) + U(s-1) - 2U(s-2))x(t-s)ds \qquad (2)$$

where $U(t)$ is the CT unit step signal.

1. [15%] Find out the impulse response $h(t)$ of this system, and plot it for the range of $-4 \le t \le 4$.

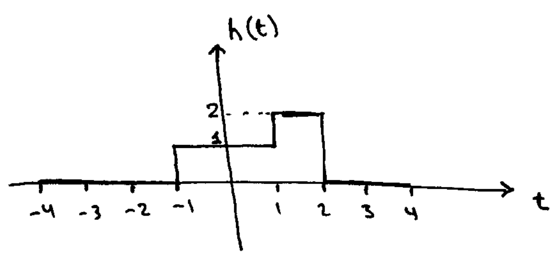
   Hint 1: If you do not know the answer to this sub-question, please write down the definition of "impulse response". You will receive 2 points if your answer is correct.

$$h(t) = \int_{-\infty}^{\infty} [u(s+1) + u(s-1) - 2u(s-2)]\,\delta(t-s)\,ds$$

$$= u(t+1) + u(t-1) - 2u(t-2) \int_{-\infty}^{\infty} \delta(t-s)\,ds$$

$$= u(t+1) + u(t-1) - 2u(t-2)$$

$$= \begin{cases} 1 & , \ -1 < t < 1 \\ 2 & , \ 1 < t < 2 \\ 0 & , \ \text{otherwise} \end{cases}$$



Question 3: [14%, Work-out question, Learning Objectives 1, 2, and 3]

Consider $x(t) = \cos(0.6\pi t + \frac{\pi}{4}) + \sin(0.3\pi t)$. Find the Fourier series representation of $x(t)$.

$$T_1 = \frac{2\pi}{0.6\pi} = \frac{10}{3} \qquad T_2 = \frac{2\pi}{0.3\pi} = \frac{20}{3}$$

$$T = \mathrm{lcm}\left(\frac{10}{3}, \frac{20}{3}\right) = \frac{20}{3}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{3\pi}{10}$$

$$x(t) = \frac{e^{j(0.6\pi t + \frac{\pi}{4})} + e^{-j(0.6\pi t + \frac{\pi}{4})}}{2} + \frac{e^{j0.3\pi t} - e^{-j0.3\pi t}}{2j}$$

$$= \frac{e^{j\frac{\pi}{4}}}{2} e^{j2\left(\frac{3\pi}{10}\right)t} + \frac{e^{-j\frac{\pi}{4}}}{2} e^{-j2\left(\frac{3\pi}{10}\right)t} + \frac{1}{2j} e^{j\left(\frac{3\pi}{10}\right)t} - \frac{1}{2j} e^{-j\left(\frac{3\pi}{10}t\right)}$$

$$a_1 = \frac{1}{2j}$$

$$a_{-1} = \frac{-1}{2j}$$

$$a_2 = \frac{e^{j\frac{\pi}{4}}}{2}$$

$$a_{-2} = \frac{e^{-j\frac{\pi}{4}}}{2}$$

$a_k = 0$ for all other k

Question 4: [15%, Work-out question, Learning Objectives 1, 2, 3, and 5]
Consider a DT LTI system with impulse response

$$h[n] = \begin{cases} e^{-(n-101)} & \text{if } 101 \le n \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Find out the output $y[n]$ when the input $x[n]$ is

$$x[n] = e^{jn} + e^{j(1.1\pi n)} \tag{4}$$

Hint 1: You may need to use the following formulas: If $r \neq 1$, then we have

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}. \tag{5}$$

If $|r| < 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}. \tag{6}$$

If $|r| \geq 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} \text{ does not exist.} \tag{7}$$

Hint 2: You can leave your answer to be something like $\frac{e^{j2n}(1+j)}{(2-j)}$. There is no need to further simplify the expression.

$y[n] = x[n] \cdot H(j\omega)$

$H(j\omega) = \sum_{n=-\infty}^{\infty} h[n] \, e^{-j\omega n}$

$= \sum_{n=101}^{\infty} e^{-(n-101)} e^{-j\omega n} = \sum_{n=101}^{\infty} e^{101} e^{-(1+j\omega)n}$ let $K = n - 100$, $n = K + 100$

$= \sum_{K=1}^{\infty} e^{101} e^{-(1+j\omega)(K+100)} = e^{1-100j\omega} \sum_{K=1}^{\infty} \left(e^{-(1+j\omega)}\right)^{K}$

$= e^{1-100j\omega} \left(e^{-(1+j\omega)}\right)^{1} \sum_{K=1}^{\infty} \left(e^{-(1+j\omega)}\right)^{K-1}$

$= e^{-101j\omega} \dfrac{1}{1 - e^{-(1+j\omega)}}$

$$y[n] = \frac{e^{jn} \cdot e^{-101j}}{1 - e^{-(1+j)}} + \frac{e^{j(1.1\pi n)} \; e^{-101j(1.1\pi)}}{1 - e^{-(1+1.1\pi j)}}$$

$$\boxed{y[n] = \frac{e^{j(n-101)}}{1 - e^{-(1+j)}} + \frac{e^{1.1\pi j(n-101)}}{1 - e^{-(1+1.1\pi j)}}}$$

OR $y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$

$$= \sum_{k=101}^{\infty} e^{-(k-101)} \left(e^{j(n-k)} + e^{j(1.1\pi(n-k))} \right)$$

$$= \sum_{k=101}^{\infty} e^{101+jn} \, e^{-k(j+1)} + \sum_{k=101}^{\infty} e^{101+j1.1\pi n} \, e^{-k(1+1.1\pi j)}$$

let $m = k - 100$

$k = m + 100$

$$= e^{101+jn} \sum_{m=1}^{\infty} e^{-(m+100)(j+1)} + e^{101+j1.1\pi n} \sum_{m=1}^{\infty} e^{-(m+100)(1+1.1\pi j)}$$

$$= e^{1+j(n-100)} \, e^{-(j+1)} \frac{1}{1 - e^{-(j+1)}} + e^{1+j1.1\pi(n-100)} \, e^{-(1+1.1\pi j)} \frac{1}{1 - e^{-(1+1.1\pi j)}}$$

$$= \frac{e^{j(n-101)}}{1 - e^{-(j+1)}} + \frac{e^{j1.1\pi j(n-101)}}{1 - e^{-(1+j1.1\pi)}}$$

Question 5: [25%, Work-out question, Learning Objective 4]

1. [4%] Consider a CT signal

$$y(t) = \begin{cases} e^{-|t|} & \text{if } |t| < 2 \\ y(t) \text{ is periodic with period 4} & \end{cases}. \tag{8}$$

   Plot $y(t)$ for the range of $-4 \leq t \leq 4$.

2. [16%] Continue from the previous sub-question. Find the Fourier series representation of $y(t)$.

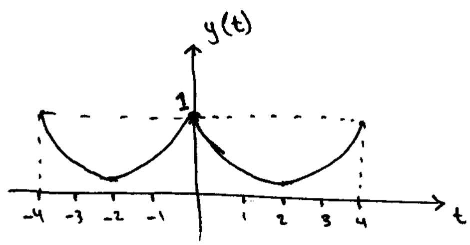
3. [5%] Consider another CT signal

$$z(t) = \begin{cases} e^{t} & \text{if } -2 \leq t < 0 \\ e^{-t} & \text{if } 0 \leq t < 2 \\ z(t) \text{ is periodic with period 4} & \end{cases}. \tag{9}$$

   Suppose the Fourier series coefficients of $z(t)$ are denoted as c_k. Find the values of c_0 and c_1.

   Hint: If you do not know the answer to the previous sub-question, you can assume that the Fourier series coefficients of $y(t)$ are $b_0 = \frac{2T_1}{T}$ and $b_k = \frac{\sin(k\frac{2\pi}{T}T_1)}{\pi k}$ with $T_1 = 0.4$ and $T = 3$. Using the values of b_k will help you find the values of c_0 and c_1. If your answer is correct, you will receive full credit for this sub-question.

1)



2)

$$a_k = \frac{1}{4}\int_{-2}^{2} e^{-|t|}\, e^{-jk\left(\frac{2\pi t}{4}\right)}\, dt$$

$$= \frac{1}{4}\int_{-2}^{0} e^{t}\, e^{-jk\frac{\pi}{2}t}\, dt + \frac{1}{4}\int_{0}^{2} e^{-t}\, e^{-jk\frac{\pi}{2}t}\, dt$$

$$= \frac{1}{4(1-jk\frac{\pi}{2})} e^{(1-jk\frac{\pi}{2})t}\Big]_{-2}^{0} + \frac{-1}{4(1+jk\frac{\pi}{2})} e^{-(1+jk\frac{\pi}{2})t}\Big]_{0}^{2}$$

$$a_k = \frac{1}{4-jk2\pi}\left[1 - e^{-(1-jk\frac{\pi}{2})2}\right] - \frac{1}{4+jk2\pi}\left[e^{-(1+jk\frac{\pi}{2})2} - 1\right]$$

$$= \frac{1 - e^{-2}e^{jk\pi}}{4-jk2\pi} + \frac{1 - e^{-2}e^{-jk\pi}}{4+jk2\pi}$$

$$\boxed{a_k = \frac{2(1 - e^{-2}(-1)^k)}{4+\pi^2k^2}}$$

$$a_0 = \frac{1}{4}\int_{-2}^{0} e^{t}\,dt + \frac{1}{4}\int_{0}^{2} e^{-t}\,dt$$

$$= \frac{1-e^{-2}}{4} - \frac{e^{-2}-1}{4} = \frac{1-e^{-2}}{2}$$

3) $z(t) = y(t)$

$$c_0 = a_0 = \frac{1-e^{-2}}{2}$$

$$c_1 = a_1 = \frac{2(1+e^{-2})}{4+\pi^2}$$

Question 6: [20%, Multiple Choices, Learning Objective 1]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \begin{cases} \int_{t-1}^{t} x_1(s)ds & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases} \tag{10}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = e^{-\int_{s=0}^{|x[n]|} s^2 ds} \cos(2n-1) \tag{11}$$

Answer the following questions

1. [4%, Learning Objective 1] Is System 1 memoryless? Is System 2 memoryless?
2. [4%, Learning Objective 1] Is System 1 causal? Is System 2 causal?
3. [4%, Learning Objective 1] Is System 1 stable? Is System 2 stable?
4. [4%, Learning Objective 1] Is System 1 linear? Is System 2 linear?
5. [4%, Learning Objective 1] Is System 1 time-invariant? Is System 2 time-invariant?

	System 1	System 2
1)	Has memory	Memoryless
2)	Causal	Causal
3)	Stable	Stable
4)	Linear	Non-linear
5)	Time Variant	Time Variant