Final Exam of ECE301, Section 3 (CRN 17101-003)

8–10am, Wednesday, December 13, 2017, Hiler Thtr.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**

2. This is a closed book exam.

3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have two hours to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.

4. Use the back of each page for rough work.

5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [21.5%, Work-out question]

1. [1%] What does the acronym AM-SSB stand for?

2. [1%] What does the acronym FDM stand for when referring to the technique of simultaneously broadcasting multiple AM signals from the same antenna tower?

Prof. Wang wanted to transmit an AM-SSB signal. To that end, he wrote the following MATLAB code.

```
% Initialialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read two different .wav files
[x1, f_sample, N]=audioread('x1.wav');
x1=x1';
[x2, f_sample, N]=audioread('x2.wav');
x2=x2';

% Step 0: Initialize several parameters
W_1=????;
W_2=pi*5000;
W_3=pi*8000;
W_4=pi*2000;
W_5=pi*3000;
W_6=????;


% Step 1: Make the signals band-limited.
h=1/(pi*t).*(sin(W_1*t));
x1_new=ece301conv(x1, h);
x2_new=ece301conv(x2, h);

% Step 2: Multiply x1_new and x2_new with a sinusoidal wave.
x1_h=x1_new.*cos(W_2*t);
x2_h=x2_new.*sin(W_3*t);

% Step 3: Keep one of the two side bands
h_one=1/(pi*t).*(2*sin(W_4*t)).*(cos(W_5*t));
h_two=1/(pi*t).*(2*sin(W_4*t)).*(cos(W_6*t));
x1_sb=ece301conv(x1_h, h_one);
x2_sb=ece301conv(x2_h, h_two);
```

```
% Step 4: Create the transmitted signal
y=x1_sb+x2_sb;
audiowrite('y.wav', y, f_sample);
```

3. [1.5%] What is the carrier frequency (Hz) of the signal x2_new?

4. [1.5%] For the first signal x1_new, is this AM-SSB transmitting an upper-side-band signal or a lower-side-band signal?

5. [1.5%] What should the value of W_6 be in the MATLAB code, if we decide to use an upper-side-band transmission for the second signal x2_new?

6. [1.5%] What should the value of W_6 be in the MATLAB code, if we decide to use a lower-side-band transmission for the second signal x2_new?

7. [2%] Continue from the previous sub-question. Suppose lower-side-band transmission is used for the second signal x2_new. To ensure that the receiver side can have the best possible quality, it is important for the transmitter to choose the largest W_1 value when possible. What is the largest W_1 value that can be used without significantly degrading the quality of any of the two transmitted signals?

Knowing that Prof. Wang decided to use a lower-side-band transmission for the second signal x2_new and choose the W_1 value to be $W_1 = 2000 \times \pi$. He then used the above code to generate the "y.wav" file, a student tried to demodulate the output waveform "y.wav" by the following code.

```
% Initialization
duration=8;
f_sample=44100;
t=(((0-4)*f_sample+0.5):((duration-4)*f_sample-0.5))/f_sample;

% Read the .wav files
[y, f_sample, N]=audioread('y.wav');
y=y';

% Initialize several parameters
W_8=????;
W_9=????;
W_10=????;
W_11=????;
W_12=????;


% Create the low-pass filter.
h_M=1/(pi*t).*(sin(W_8*t));

% demodulate signal 1
h_BPF1=1/(pi*t).*(sin(W_9*t));
y1_BPF=ece301conv(y,h_BPF1);
y1=2*y1_BPF.*cos(pi*5000*t);
x1_hat=ece301conv(y1,h_M);

sound(x1_hat,f_sample)

% demodulate signal 2
h_BPF2=1/(pi*t).*(sin(W_10*t))-1/(pi*t).*(sin(W_11*t));
y2_BPF=ece301conv(y,h_BPF2);
y2=2*y2_BPF.*cos(pi*W_12*t);
x2_hat=ece301conv(y2,h_M);

sound(x2_hat,f_sample)
```

8. [7.5%] Continue from the previous questions. What should the values of W_8 to W_12 be in the MATLAB code? When answering this question, please assume that

the second radio x2_new is transmitted using the *lower side-band* and $W_1 = 2000 \times \pi$.

9. [2%] It turns out that the above MATLAB code is not written correctly and neither signal x1_new nor signal x2_new can be correctly demodulated. Please use 2 to 3 sentences to (i) what kind of problem does x1_new have, i.e., how does the problem impact the sound quality of "sound(x1_hat,f_sample)"? (ii) how can the MATLAB code be corrected so that the playback/demodulation can performed successfully?

10. [2%] Please use 2 to 3 sentences to (i) what kind of problem does x2_new have, i.e., how does the problem impact the sound quality of "sound(x2_hat,f_sample)"? (ii) how can the MATLAB code be corrected so that the playback/demodulation can performed successfully?

    Hint: If you do not know the answers of Q1.3 to Q1.10, please simply draw the AMSSB modulation and demodulation diagrams and mark carefully all the parameter values. You will receive 12 points for Q1.3 to Q1.10.

Question 2: [11.5%, Work-out question]

1. [1.5%] Consider a continuous time signal $x(t)$

$$x(t) = \frac{\sin(2\pi t)}{\pi t}. \tag{1}$$

   Plot the CTFT $X(j\omega)$ of $x(t)$ for the range of $-4\pi \leq \omega \leq 4\pi$.

2. [2%] We then construct $y(t)$ by

$$y(t) = 2\cos(100\pi t). \tag{2}$$

   Plot the CTFT $Y(j\omega)$ of $y(t)$ for the range of $-104\pi \leq \omega \leq 104\pi$.

3. [3%] Finally we construct $z(t)$ by

$$z(t) = y(t) * \frac{\sin(101\pi t)}{\pi t} \tag{3}$$

   Plot the CTFT $Z(j\omega)$ of $z(t)$ for the range of $-104\pi \leq \omega \leq 104\pi$.

   Hint: If you do not know how to solve this question, you can solve the following alternative question instead. You will receive 2 points if your answer is correct.

   Suppose $h(t) = \frac{\sin(100t)}{t} * \frac{\sin(20t)}{\pi t}$. Find the CTFT $H(j\omega)$.

4. [5%] Continue from the previous sub-question. Plot $z(t)$ for the range of $-5 < t < 5$.

   Hint: If you do not know how to solve this question, please write down what is the definition of *AM asynchronous demodulation* and give a detailed example how to use AM asynchronous demodulation to demodulate an AM signal. If your answer is correct, you will receive 3 points for this sub-question.

Question 3: [13%, Work-out question]

1. [2%] Consider a continuous time signal

$$x(t) = \begin{cases} 2t & \text{if } 0.001 < t < 1.001 \\ 2 & \text{if } 1.001 < t < 2.001 \\ \text{periodic with period 2} \end{cases} \qquad (4)$$

   Plot $x(t)$ for the range of $-4 < t < 4$.

2. [4%] We sample $x(t)$ with the sampling frequency 1.5Hz and denote the sampled values by $x[n]$. Plot $x[n]$ for the range of $-5 \le n \le 5$.

3. [3%] We use $x_{\text{ZOH}}(t)$ to represent the reconstructed signal using "zero-order hold". Plot $x_{\text{ZOH}}(t)$ for the range of $-4 < t < 4$.

   Hint: if you do not know the answer of $x[n]$, you can assume that $x[n] = 2(-1)^n + 1$ and the sampling frequency is 1.5Hz. You will receive full points if your answer is correct.

4. [4%] Consider another continuous time signal $y(t)$ and again we sample $y(t)$ with sampling frequency 1.5Hz. Suppose the resulting $y[n] = 2\delta[n - 5]$. Let $y_{\text{opt}}(t)$ denote the optimal band-limited reconstruction of $y(t)$. Plot $y_{\text{opt}}(t)$ for the range of $-6 < t < 6$.

Question 4: [11%, Work-out question]

Consider a signal $x(t) = \sin(1.25\pi t)$. Let $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$. Define $x_p(t) = x(t) \cdot p(t)$, i.e., $x_p(t)$ is the impulse train sampled signal.

1. [1%] What is the sampling frequency (Hz) of the impulse train sampled signal $x_p(t)$?

2. [4.5%] What is the CTFT $X_p(j\omega)$ of $x_p(t)$? Plot $X_p(j\omega)$ for the range of $-\pi < \omega < \pi$.

   Hint: If you do not know the answer to this question, you should find CTFT $X(j\omega)$ of $x(t)$ instead. You will receive 2 points if your answer is correct.

3. [5.5%] We reconstruct $x(t)$ by the following operation:

$$\hat{x}(t) = 2x_p(t) * \frac{\sin(0.5\pi t)}{\pi t}. \tag{5}$$

   What is the expression of $\hat{x}(t)$?

   Hint: If you do not know the answer to this question, please write down in details (i) What is the *sampling theorem*? (ii) What is the *Nyquist frequency*? You will receive 3 points if your answers are correct.

Question 5: [10%, Work-out question]

Consider the following continuous time signals

$$x(t) = \begin{cases} e^{2t} & \text{if } t < -1 \\ 0 & \text{if } -1 \leq t < 1 \\ e^{-3t} & \text{if } 1 \leq t \end{cases} \tag{6}$$

$$y(t) = e^{-(t-4)}\mathcal{U}(t-4). \tag{7}$$

Find the expression of $z(t) = x(t) * y(t)$. There is no need to plot $z(t)$. A case-based expression would suffice.

Question 6: [11%, Work-out question]
   Consider the following discrete-time signals.

$$x[n] = \frac{\sin\left(0.75\pi n\right)}{\pi n} \qquad (8)$$

1. [3%] Plot DTFT $X(e^{j\omega})$ for the range of $-3\pi < \omega < 3\pi$.

2. [3%] Let $y[n] = x[n](-1)^n$. Plot DTFT $Y(e^{j\omega})$ for the range of $-3\pi < \omega < 3\pi$.

   Hint: if you do not know what is $X(e^{j\omega})$, you can assume $X(e^{j\omega}) = \cos(\omega) + 1$. You will receive full points if your answer is correct.

3. [5%] Let $z[n] = x[n] * y[n]$. Find the expression of $z[n]$.

   Hint: if you do not know how to solve this question, you can solve the following question instead.

   $x(t) = \cos(t) + \sin(2t)$, $y(t) = \frac{\sin(1.5t)}{1.5t}$, and $z(t) = x(t) * y(t)$. Find the expression of $z(t)$. You will receive 3 points if your answer is correct.

Question 7: [7%, Work-out question]
Consider the following discrete time system.

$$y[n] = \begin{cases} \sum_{k=n}^{n+5} x[k] & \text{if } n \leq -8 \\ \sum_{k=n-5}^{n} x[k] & \text{if } -7 \leq n \end{cases}. \tag{9}$$

1. [4.5%] Find the expression of the impulse response $h[n]$ of this system.

2. [2.5%] Is the above system causal? This is not a yes-no question. Please carefully explain your answer in 1 to 3 sentences.

Question 8: [15%, Multiple-choice question] Consider two signals

$$h_1(t) = |e^{e^{jt}}| \tag{10}$$

and

$$h_2[n] = (\cos(0.25\pi n^3) + e^{j\pi n} - e^{-j\pi n}) \cdot (\mathcal{U}[n + 99] - \mathcal{U}[n - 100]) \tag{11}$$

Hint: you should treat $h_1(t)$ as $|e^{x(t)}|$ where $x(t) = e^{jt}$.

1. [1.25%] Is $h_1(t)$ periodic?

2. [1.25%] Is $h_2[n]$ periodic?

3. [1.25%] Is $h_1(t)$ even or odd or neither?

4. [1.25%] Is $h_2[n]$ even or odd or neither?

5. [1.25%] Is $h_1(t)$ of finite energy?

6. [1.25%] Is $h_2[n]$ of finite energy?

Suppose the above two signals are also the impulse responses of two LTI systems: System 1 and System 2, respectively.

1. [1.25%] Is System 1 memoryless?

2. [1.25%] Is System 2 memoryless?

3. [1.25%] Is System 1 causal?

4. [1.25%] Is System 2 causal?

5. [1.25%] Is System 1 stable?

6. [1.25%] Is System 2 stable?

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(j\omega) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$

TABLE 3.1   PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$ } Periodic with period T and	a_k
		$y(t)$ } fundamental frequency $\omega_0 = 2\pi/T$	b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau) y(t - \tau) d\tau$	$T a_k b_k$
Multiplication	3.5.5	$x(t) y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\dfrac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\dfrac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\dfrac{1}{jk\omega_0}\right) a_k = \left(\dfrac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ |a_k| = |a_{-k}| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} \quad [x(t)\text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} \quad [x(t)\text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t - 1) - 1/2.$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2   PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$x[n]$ } Periodic with period N and $y[n]$ } fundamental frequency $\omega_0 = 2\pi/N$	a_k } Periodic with b_k } period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a_{-k}^*
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\dfrac{1}{m}a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\displaystyle\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\displaystyle\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\displaystyle\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\dfrac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a_{-k}^* \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ |a_k| = |a_{-k}| \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} \quad [\text{x[n] real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} \quad [\text{x[n] real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

TABLE 4.1   PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$	$X(j\omega)$
		$y(t)$	$Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\dfrac{1}{\|a\|}X\left(\dfrac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\dfrac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\displaystyle\int_{-\infty}^{t}x(t)dt$	$\dfrac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\dfrac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$ $x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

4.3.7   Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$

TABLE 4.2   BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\displaystyle\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$\displaystyle 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0,\quad$ otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0,\quad$ otherwise
$\sin \omega_0 t$	$\dfrac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0,\quad$ otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1,\quad a_k = 0,\ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\displaystyle\sum_{k=-\infty}^{+\infty} \dfrac{2\sin k\omega_0 T_1}{k}\,\delta(\omega - k\omega_0)$	$\dfrac{\omega_0 T_1}{\pi}\,\mathrm{sinc}\left(\dfrac{k\omega_0 T_1}{\pi}\right) = \dfrac{\sin k\omega_0 T_1}{k\pi}$
$\displaystyle\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\dfrac{2\pi}{T} \displaystyle\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \dfrac{2\pi k}{T}\right)$	$a_k = \dfrac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\dfrac{2\sin \omega T_1}{\omega}$	—
$\dfrac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\dfrac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t),\ \mathcal{R}e\{a\} > 0$	$\dfrac{1}{a + j\omega}$	—
$te^{-at}u(t),\ \mathcal{R}e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^2}$	—
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t),$ $\mathcal{R}e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^n}$	—

TABLE 5.1   PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$	$X(e^{j\omega})\}$ periodic with
		$y[n]$	$Y(e^{j\omega})\}$ period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n - 1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\displaystyle\sum_{k=-\infty}^{n} x[k]$	$\dfrac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\displaystyle\sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{Re}\{X(e^{j\omega})\} = \mathcal{Re}\{X(e^{-j\omega})\} \\ \mathcal{Im}\{X(e^{j\omega})\} = -\mathcal{Im}\{X(e^{-j\omega})\} \\ |X(e^{j\omega})| = |X(e^{-j\omega})| \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{Ev}\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{Od}\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{Re}\{X(e^{j\omega})\}$ $j\mathcal{Im}\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

a duality relationship between the discrete-time Fourier transform and the continuous-time Fourier series. This relation is discussed in Section 5.7.2.

5.7.1 Duality in the Discrete-Time Fourier Series

Since the Fourier series coefficients a_k of a periodic signal $x[n]$ are themselves a periodic sequence, we can expand the sequence a_k in a Fourier series. The duality property for discrete-time Fourier series implies that the Fourier series coefficients for the periodic sequence a_k are the values of $(1/N)x[-n]$ (i.e., are proportional to the values of the original

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\displaystyle\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$\displaystyle 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$\displaystyle 2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \dfrac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\dfrac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos\omega_0 n$	$\displaystyle \pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \dfrac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\dfrac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin\omega_0 n$	$\displaystyle \frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \dfrac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\dfrac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$\displaystyle 2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & N_1 < \|n\| \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$\displaystyle 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \dfrac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \ k \neq 0, \pm N, \pm 2N, \dots$ $a_k = \dfrac{2N_1 + 1}{N}, \ k = 0, \pm N, \pm 2N, \dots$
$\displaystyle\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\displaystyle \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \dfrac{1}{N}$ for all k
$a^n u[n], \quad \|a\| < 1$	$\dfrac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & \|n\| > N_1 \end{cases}$	$\dfrac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\dfrac{\sin Wn}{\pi n} = \dfrac{W}{\pi} \operatorname{sinc}\left(\dfrac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \|\omega\| \le W \\ 0, & W < \|\omega\| \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\dfrac{1}{1 - e^{-j\omega}} + \displaystyle\sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n + 1)a^n u[n], \quad \|a\| < 1$	$\dfrac{1}{(1 - ae^{-j\omega})^2}$	—
$\dfrac{(n + r - 1)!}{n!(r - 1)!} a^n u[n], \quad \|a\| < 1$	$\dfrac{1}{(1 - ae^{-j\omega})^r}$	—

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $|a|R$ = the set of points $\{|a|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $|z| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $|z| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R
10.5.9		Initial Value Theorem		

If $x[n] = 0$ for $n < 0$, then
$$x[0] = \lim_{z\to\infty} X(z)$$

TABLE 10.2   SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1 - z^{-1}}$	$|z| > 1$
3. $-u[-n-1]$	$\dfrac{1}{1 - z^{-1}}$	$|z| < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $\alpha^n u[n]$	$\dfrac{1}{1 - \alpha z^{-1}}$	$|z| > |\alpha|$
6. $-\alpha^n u[-n-1]$	$\dfrac{1}{1 - \alpha z^{-1}}$	$|z| < |\alpha|$
7. $n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$|z| > |\alpha|$
8. $-n\alpha^n u[-n-1]$	$\dfrac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$	$|z| < |\alpha|$
9. $[\cos \omega_0 n]u[n]$	$\dfrac{1 - [\cos \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$|z| > 1$
10. $[\sin \omega_0 n]u[n]$	$\dfrac{[\sin \omega_0]z^{-1}}{1 - [2\cos \omega_0]z^{-1} + z^{-2}}$	$|z| > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\dfrac{1 - [r\cos \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$|z| > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\dfrac{[r\sin \omega_0]z^{-1}}{1 - [2r\cos \omega_0]z^{-1} + r^2 z^{-2}}$	$|z| > r$

10.7.1 Causality

A causal LTI system has an impulse response $h[n]$ that is zero for $n < 0$, and therefore is right-sided. From Property 4 in Section 10.2 we then know that the ROC of $H(z)$ is the exterior of a circle in the z-plane. For some systems, e.g., if $h[n] = \delta[n]$, so that $H(z) = 1$, the ROC can extend all the way in to and possibly include the origin. Also, in general, for a right-sided impulse response, the ROC may or may not include infinity. For example, if $h[n] = \delta[n + 1]$, then $H(z) = z$, which has a pole at infinity. However, as we saw in Property 8 in Section 10.2, for a causal system the power series

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$$

does not include any positive powers of z. Consequently, the ROC includes infinity. Summarizing, we have the follow principle:

A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, including infinity.