**Midterm #2 of ECE301, Section 2**

8–9pm, Thursday, October 13, 2016, ME 1061.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

*Question 1:* [32%, Work-out question, Learning Objectives 1, 2, 3, and 4]

1. [2%] What is the definition of "impulse responses $h(t)/h[n]$"?

Consider three CT-LTI systems with different impulse responses:

$$\begin{aligned} h_1(t) &= \delta(t + 2.5) \\ h_2(t) &= e^{-|t-2|} \\ h_3(t) &= e^{-t}\mathcal{U}(2t - \pi) \end{aligned} \tag{1}$$

2. [3%] Is the first system causal? Is the second system causal? Is the third system causal? These are yes/no questions and there is no need to justify your answers.

3. [2%] Is the second system stable? Is the third system stable? These are yes/no questions and there is no need to justify your answers.

Suppose we serially concatenate the first two systems together. Namely, for any input $x(t)$, we first pass it through the first system with impulse response $h_1(t)$ and denote the (intermediate) output signal by $z(t)$. Then we pass $z(t)$ through the second system with impulse response $h_2(t)$ and denote the final output by $y(t)$.

4. [10%] Denote the impulse response of the above serially concatenated system by $h_{1\to 2}(t)$. Find the expression of $h_{1\to 2}(t)$.

   Hint 1: There are several different ways to solve this problem. One way is to treat $h_1(t)$ as an input and use the fact that we are focusing on a Linear Time-Invariant system.

Suppose we serially concatenate the *last* two systems together. Namely, for any input $x(t)$, we first pass it through the second system with impulse response $h_2(t)$ and denote the (intermediate) output signal by $z(t)$. Then we pass $z(t)$ through the third system with impulse response $h_3(t)$ and denote the final output by $y(t)$.

5. [15%] Denote the impulse response of the above serially concatenated system by $h_{2\to 3}(t)$. Find the expression of $h_{2\to 3}(t)$.

   Hint 2: You should at least figure out how to find the "cases" of this problem. If you can find the cases and the corresponding integrals correctly (without actually solving the integral), you will still get 8 points for this question.

*Question 2:* [14%, Work-out question, Learning Objectives 2, 3, 4, and 5] Consider a CT-LTI system with impulse response:

$$h(t) = e^{t}\mathcal{U}(-t). \tag{2}$$

If the input signal is $x(t) = e^{jt} + \cos(\sqrt{3}t - 1)$, find the corresponding output $y(t)$.

*Question 3:* [24%, Work-out question, Learning Objectives 3 and 4]

1. [8%] Consider a CT signal $x(t) = \cos(\frac{\pi}{3}t)$. Plot $x(t)$ for the range of $-6 < t < 6$ and find the CTFS coefficients $a_k$ of $x(t)$.

2. [8%] Consider another CT signal $y(t)$:

$$y(t) = \begin{cases} 2 & \text{if } -1.5 \leq t < 1.5 \\ 0 & \text{if } 1.5 \leq t < 4.5 \\ \text{periodic with period } 6 \end{cases} . \quad (3)$$

Plot $y(t)$ for the range of $-6 < t < 6$; Find the CTFS coefficients $b_k$ of $y(t)$; and find the values of $b_9$ and $b_{-4}$.

Hint: You can use the following fact directly. For a rectangular waveform with period $T$, amplitude 1, and duty cycle $\frac{2T_1}{T}$, the CTFS coefficients are

$$a_0 = \frac{2T_1}{T},\ a_k = \frac{\sin\left(k\frac{2\pi}{T}T_1\right)}{k\pi} \text{ for all } k \neq 0. \quad (4)$$

3. [8%] Consider another CT signal $z(t)$:

$$z(t) = \begin{cases} \cos(\frac{\pi}{3}t) & \text{if } -1.5 \leq t < 1.5 \\ 0 & \text{if } 1.5 \leq |t| < 3 \\ \text{periodic with period } 6 \end{cases} . \quad (5)$$

Plot $z(t)$ for the range of $-6 < t < 6$ and find the CTFS coefficients $c_k$ of $z(t)$ in terms of $b_k$.

Hint 1: If you do not know the answer of $a_k$ in the first sub-question, you can assume $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and all other $a_k = 0$. If you use the "assumed $a_k$ values" and solve the question correctly, you will still get full credit for this question.

Hint 2: If you do not know how to express your solution only in terms of $b_k$, you can write $c_k$ as a function of $a_k$ and $b_k$. You will still receive 4 points if your answer is correct.

*Question 4:* [10%, Work-out question, Learning Objective 4] Consider a DT signal

$$x[n] = \begin{cases} e^{jn} & \text{if } 0 \le n < 50 \\ -1 + j & \text{if } 50 \le n < 100 \\ 0 & \text{if } 100 \le n < 200 \\ \text{periodic with period } 200 & \end{cases}. \tag{6}$$

Denote the DTFS of $x[n]$ by $(a_k, \frac{2\pi}{200})$. Find the value of the following summation:

$$\sum_{k=0}^{199} |a_k|^2 \tag{7}$$

*Question 5:* [20%, Multiple Choices] The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

**System 1:** When the input is $x_1(t)$, the output is

$$y_1(t) = \int_{t-1}^{t} |x_1(s)| ds \tag{8}$$

**System 2:** When the input is $x_2[n]$, the output is

$$y_2[n] = \begin{cases} \sum_{k=\frac{n}{3}}^{\frac{n}{3}+100} e^{x_2[k]} & n \text{ is a multiple of } 3 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Answer the following questions

1. [4%, Learning Objective 1] Is System 1 memoryless? Is System 2 memoryless?
2. [4%, Learning Objective 1] Is System 1 causal? Is System 2 causal?
3. [4%, Learning Objective 1] Is System 1 stable? Is System 2 stable?
4. [4%, Learning Objective 1] Is System 1 linear? Is System 2 linear?
5. [4%, Learning Objective 1] Is System 1 time-invariant? Is System 2 time-invariant?

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(j\omega)e^{j\omega n}d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi}e^{\sigma t}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{10}$$

Z transform

$$x[n] = r^n\mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{12}$$

**TABLE 3.1** PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

| Property | Section | Periodic Signal | Fourier Series Coefficients |
|---|---|---|---|
| | | $x(t)$, $y(t)$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$ | $a_k$<br>$b_k$ |
| Linearity | 3.5.1 | $Ax(t) + By(t)$ | $Aa_k + Bb_k$ |
| Time Shifting | 3.5.2 | $x(t - t_0)$ | $a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$ |
| Frequency Shifting | | $e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$ | $a_{k-M}$ |
| Conjugation | 3.5.6 | $x^*(t)$ | $a^*_{-k}$ |
| Time Reversal | 3.5.3 | $x(-t)$ | $a_{-k}$ |
| Time Scaling | 3.5.4 | $x(\alpha t), \alpha > 0$ (periodic with period $T/\alpha$) | $a_k$ |
| Periodic Convolution | | $\int_T x(\tau)y(t-\tau)d\tau$ | $Ta_k b_k$ |
| Multiplication | 3.5.5 | $x(t)y(t)$ | $\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$ |
| Differentiation | | $\frac{dx(t)}{dt}$ | $jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$ |
| Integration | | $\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$) | $\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$ |
| Conjugate Symmetry for Real Signals | 3.5.6 | $x(t)$ real | $a_k = a^*_{-k}$<br>$\Re e\{a_k\} = \Re e\{a_{-k}\}$<br>$\Im m\{a_k\} = -\Im m\{a_{-k}\}$<br>$\lvert a_k\rvert = \lvert a_{-k}\rvert$<br>$\sphericalangle a_k = -\sphericalangle a_{-k}$ |
| Real and Even Signals | 3.5.6 | $x(t)$ real and even | $a_k$ real and even |
| Real and Odd Signals | 3.5.6 | $x(t)$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | | $x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$<br>$x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$ | $\Re e\{a_k\}$<br>$j\Im m\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

## Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t-1) - 1/2.$$

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

## 3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

**TABLE 3.2** PROPERTIES OF DISCRETE-TIME FOURIER SERIES

| Property | Periodic Signal | Fourier Series Coefficients |
|---|---|---|
| | $\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period $N$ and fundamental frequency $\omega_0 = 2\pi/N$ | $\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period $N$ |
| Linearity | $Ax[n] + By[n]$ | $Aa_k + Bb_k$ |
| Time Shifting | $x[n - n_0]$ | $a_k e^{-jk(2\pi/N)n_0}$ |
| Frequency Shifting | $e^{jM(2\pi/N)n} x[n]$ | $a_{k-M}$ |
| Conjugation | $x^*[n]$ | $a^*_{-k}$ |
| Time Reversal | $x[-n]$ | $a_{-k}$ |
| Time Scaling | $x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period $mN$) | $\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$ |
| Periodic Convolution | $\sum_{r=\langle N \rangle} x[r]y[n-r]$ | $Na_k b_k$ |
| Multiplication | $x[n]y[n]$ | $\sum_{l=\langle N \rangle} a_l b_{k-l}$ |
| First Difference | $x[n] - x[n-1]$ | $(1 - e^{-jk(2\pi/N)})a_k$ |
| Running Sum | $\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$ | $\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$ |
| Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$ |
| Real and Even Signals | $x[n]$ real and even | $a_k$ real and even |
| Real and Odd Signals | $x[n]$ real and odd | $a_k$ purely imaginary and odd |
| Even-Odd Decomposition of Real Signals | $\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$ | $\Re e\{a_k\}$ <br> $j\Im m\{a_k\}$ |

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x[n] \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$$