Midterm #3 of ECE301, Section 2
8–9pm, Monday, April 6, 2015, EE 170.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solution Key

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [28%, Work-out question, Learning Objectives 1, 2, 4, and 5] Consider a periodic discrete-time signal

$$x[n] = \begin{cases} \sin\left(\frac{\pi n}{4}\right) & \text{if } 0 \le n \le 3 \\ \text{periodic with period } N = 4 \end{cases}. \tag{1}$$

1. [3%] Plot $x[n]$ for the range of $-6 \le n \le 6$.
2. [1%] Is the value $x[n] < 0$ for any n? (This is a yes/no question. No need to justify your answer).
3. [12%] Find the Fourier series representation (a_k, ω_0) of $x[n]$.

Continue from the above questions. Consider a discrete-time LTI system with impulse response

$$h[n] = 10^{-|n|}. \tag{2}$$

We use $x[n]$ as the input to the above LTI system and denote the output by $y[n]$.

4. [2%] Is $y[n]$ periodic? If so, write down the period of $y[n]$. If not, explain why $y[n]$ is not periodic by one sentence or two.
5. [10%] Express the Fourier series coefficient value b_3 of $y[n]$ in terms of the (a_k, ω_0) computed in the previous questions.

   Hint: You may need to use the summation formula:

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \quad \text{if } |r| < 1. \tag{3}$$

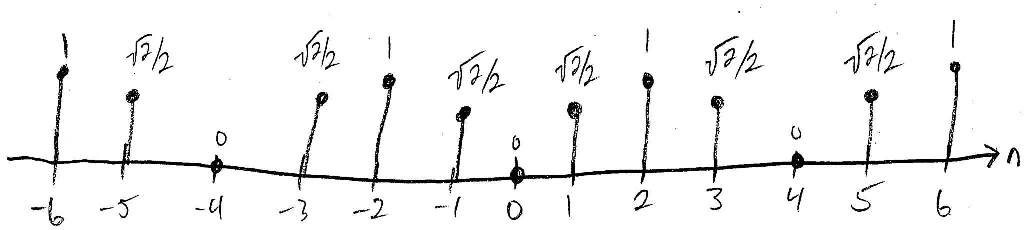
1.) $x[n] = \begin{cases} \sin\left(\frac{\pi n}{4}\right) & , 0 \le n \le 3 \\ \text{periodic with } N=4 \end{cases}$

$x[0] = \sin(0) = 0$

$x[1] = \sin(\pi/4) = \sqrt{2}/2$

$x[2] = \sin(\pi/2) = 1$

$x[3] = \sin(3\pi/4) = \sqrt{2}/2$



2.) No, $x[n]$ is always > 0

3.) DT FS $\quad a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\frac{2\pi}{N}n}$, $N=4$, $n \in \{0,1,2,3\}$

$$a_k = \frac{1}{4}\sum_{n=0}^{3} \sin\left(\frac{\pi n}{4}\right) e^{-jk\frac{2\pi}{4}n}$$

$$a_k = \frac{1}{4}\left[0 + \frac{\sqrt{2}}{2} e^{-jk\frac{\pi}{2}} + e^{-jk\pi} + \frac{\sqrt{2}}{2} e^{-jk\frac{3\pi}{2}}\right]$$

$$a_0 = \frac{1}{4}\left[\sqrt{2}+1\right]$$

$$a_1 = \frac{1}{4}\left[-\frac{\sqrt{2}}{2}j - 1 + \frac{\sqrt{2}}{2}j\right] = -\frac{1}{4}$$

$$a_2 = \frac{1}{4}\left[-\frac{\sqrt{2}}{2} + 1 - \frac{\sqrt{2}}{2}\right] = \frac{1}{4}\left[1-\sqrt{2}\right]$$

$$a_3 = \frac{1}{4}\left[\frac{\sqrt{2}}{2}j - 1 - \frac{\sqrt{2}}{2}j\right] = -\frac{1}{4}$$

$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{2}$
$N = 4$
$a_0 = \frac{1+\sqrt{2}}{4}$
$a_1 = -1/4$
$a_2 = \frac{1-\sqrt{2}}{4}$
$a_3 = -\frac{1}{4}$

4.) $y[n]$ is periodic with period $N=4$

$y[n]$ is the result of sending a sum of discrete time complex exponentials through an LTI system. We Know that an LTI system will only change the amplitude and phase of the input signal when the input is a complex exponential (sinusoid)

5.) Recall

$$\begin{array}{ccc} x[n] & \xleftrightarrow{\;*h[n]\;} & y[n] = x[n] * h[n] \\ {\scriptstyle FS}\updownarrow & & \updownarrow \\ a_K & \xleftrightarrow{\;H(e^{j\omega})\;} & b_K = a_K H(e^{jK\omega_0}) \\ \omega_0 & & \omega_0 \end{array}$$

where

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} 10^{-|n|} e^{-j\omega n} = \sum_{n=-\infty}^{0} (10e^{-j\omega})^{n} + \sum_{n=0}^{\infty} (10e^{j\omega})^{-n} - 1$$

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\frac{1}{10e^{-j\omega}}\right)^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{10e^{j\omega}}\right)^{n} - 1$$

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{10e^{-j\omega}}} + \frac{1}{1-\frac{1}{10e^{j\omega}}} - 1$$

$$b_3 = a_3 H(e^{j3\omega_0}) = a_3 H(e^{j3\frac{2\pi}{4}}) = a_3 H(e^{j\frac{3\pi}{2}}).$$

$$b_3 = a_3\left(\frac{1}{1-\frac{1}{10e^{-j\frac{3\pi}{2}}}} + \frac{1}{1-\frac{1}{10e^{j\frac{3\pi}{2}}}} - 1\right)$$

$$b_3 = a_3\left(\frac{1}{1-\frac{1}{10i}} + \frac{1}{1+\frac{1}{10i}} - 1\right) = a_3\left(\frac{10i}{10i-1} + \frac{10i}{10i+1} - 1\right)$$

$$b_3 = \left(\frac{100i^2 + 10i + 100i^2 - 10i}{100i^2 - 1} - 1\right) a_3$$

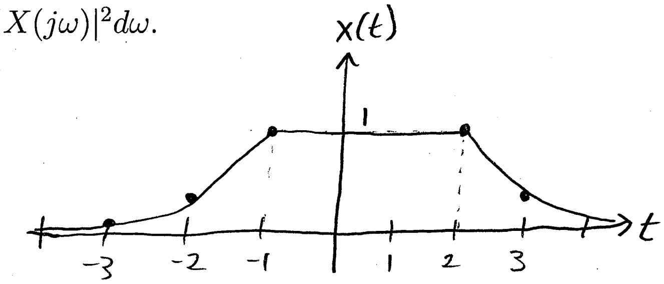
$b_3 = \dfrac{99}{101} a_3$	$N = 4$
	$\omega_0 = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

Question 2: [22%, Work-out question, Learning Objectives 1, 4, and 5]
Consider a continuous time signal

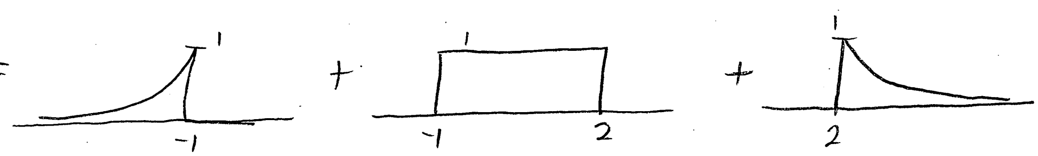
$$x(t) = \begin{cases} 1 & \text{if } -1 \le t \le 2 \\ e^{-(t-2)} & \text{if } 2 < t \\ e^{1+t} & \text{if } t < -1 \end{cases} . \tag{4}$$

1. [2%] Plot $x(t)$ for the range of $-3 \le t \le 3$.
2. [10%] Find the Fourier transform $X(j\omega)$ of $x(t)$. Hint: Tables 4.1 and 4.2 can be useful in this question.
3. [4%] Compute the value of $\int_{-\infty}^{\infty} X(j\omega)d\omega$.
4. [6%] Compute the value of $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$.

1.) $x(t) = \begin{cases} 1 & , -1 \le t \le 2 \\ e^{-(t-2)} & , 2 < t \\ e^{t+1} & , t < -1 \end{cases}$



2.) $x(t) =$



$$x(t) = e^{-(-(t+1))}u(-(t+1)) + \begin{cases} 1 & , |t-.5| < 1.5 \\ 0 & , |t-.5| > 1.5 \end{cases} + e^{-(t-2)}u(t-2)$$

First consider

$$e^{-t}u(t) \xleftrightarrow{FT} \frac{1}{1+j\omega}$$

From Table 4.1 Time Reversal

$$e^{t}u(-t) \xleftrightarrow{FT} \frac{1}{1-j\omega}$$

From Table 4.1 Time Shift

$$e^{(t+1)}u(-(t+1)) \xleftrightarrow{FT} \frac{e^{j\omega}}{1-j\omega} \quad 1.$$

Consider again

$$e^{-t}u(t) \xleftrightarrow{FT} \frac{1}{1+j\omega}$$

From Table 4.1 Time Shift

$$e^{-(t-2)}u(t-2) \xleftrightarrow{FT} \frac{e^{-j2\omega}}{1+j\omega} \quad 2.$$

Now consider

$$\begin{cases} 1 & , |t| < 1.5 \\ 0 & , |t| > 1.5 \end{cases} \xleftrightarrow{FT} \frac{2\sin(1.5\omega)}{\omega}$$

From Table 4.1 Time Shift

$$\begin{cases} 1 & , |t-\frac{1}{2}| < 1.5 \\ 0 & , |t-\frac{1}{2}| > 1.5 \end{cases} \xleftrightarrow{FT} \frac{2e^{-j\frac{\omega}{2}}\sin(1.5\omega)}{\omega} \quad 3.$$

$$X(j\omega) = 1. + 2. + 3.$$

$$\boxed{X(j\omega) = \frac{e^{j\omega}}{1-j\omega} + \frac{e^{-j2\omega}}{1+j\omega} + \frac{2e^{-j\frac{\omega}{2}}\sin(1.5\omega)}{\omega}}$$

3.) $$\int_{-\infty}^{\infty} X(j\omega)\,d\omega = \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(0)}\,d\omega = 2\pi\,x(0) = \boxed{2\pi}$$

4.) $$\int_{-\infty}^{\infty} |X(j\omega)|^2\,d\omega = 2\pi\int_{-\infty}^{\infty} |x(t)|^2\,dt$$

$$= 2\pi\left[\int_{-\infty}^{-1} (e^{t+1})^2\,dt + \int_{-1}^{2} 1^2\,dt + \int_{2}^{\infty} (e^{2-t})^2\,dt\right]$$

$$= 2\pi\left[\int_{-\infty}^{-1} e^{2+2t}\,dt + 3 + \int_{2}^{\infty} e^{4-2t}\,dt\right]$$

$$= 2\pi\left[e^{2}\int_{-\infty}^{-1} e^{2t}\,dt + 3 + e^{4}\int_{2}^{\infty} e^{-2t}\,dt\right.$$

$$= 2\pi\left[\left. e^{2}\frac{e^{2t}}{2}\right]_{-\infty}^{-1} + 3 + \left. e^{4}\frac{e^{-2t}}{-2}\right]_{2}^{\infty}\right]$$

$$= 2\pi\,\frac{e^{2}}{2}\left(e^{-2}-0\right) + 6\pi + 2\pi e^{4}\left(\frac{e^{-4}}{2}-0\right)$$

$$= \pi + 6\pi + \pi$$

$$= \boxed{8\pi}$$

Question 3: [30%, Work-out question, Learning Objectives 1, 2, 3, 4, and 5]
Consider a continuous-time LTI system with impulse response

$$h(t) = \frac{\sin(t)\sin(2t)}{t^2}. \tag{5}$$

1. [15%] Find the *frequency response* of the above system.

Let $y(t)$ denote the output when the input is $x(t) = 2 + \cos(2t) + \sin(4t + 0.25\pi)$.

2. [15%] Find the Fourier transform $Y(j\omega)$ of $y(t)$. Hint: If you do not know the answer to the previous question, you can assume that

$$H(j\omega) = e^{-2|\omega|-1}. \tag{6}$$

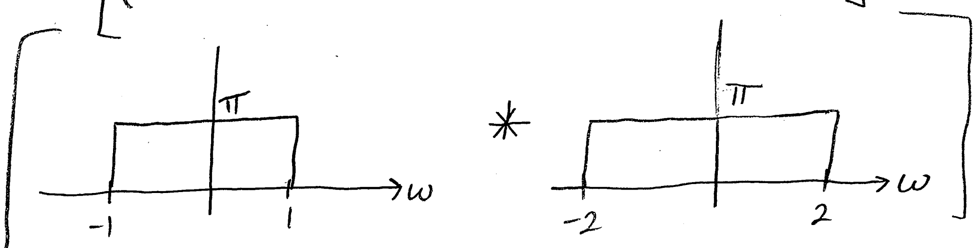
You will get full credit (15 points) if your answer is correct.

1.) $h(t) = \frac{\sin(t)}{t} \cdot \frac{\sin(2t)}{t}$

• multiplication in time is convolution in frequency

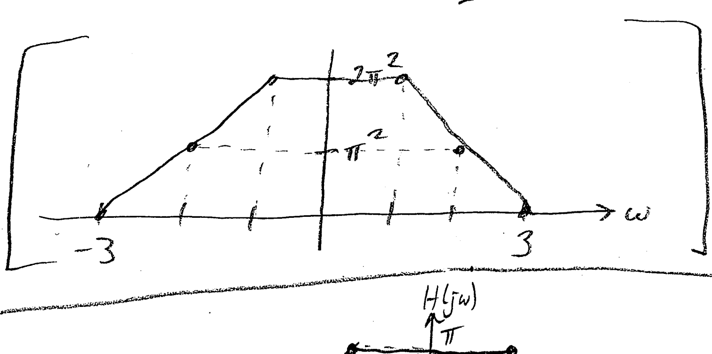
$$H(j\omega) = \left[\begin{cases} \pi, & |\omega| < 1 \\ 0, & |\omega| > 1 \end{cases} * \begin{cases} \pi, & |\omega| < 2 \\ 0, & |\omega| > 2 \end{cases} \right] \cdot \frac{1}{2\pi}$$

$H(j\omega) =$



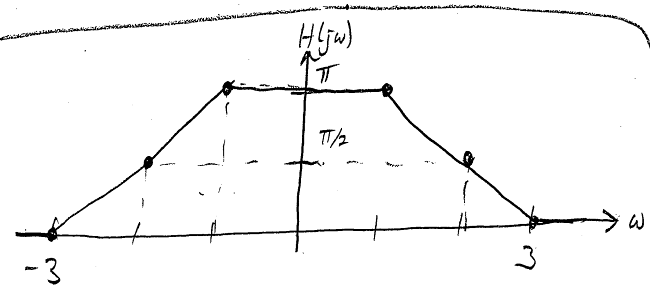
$\cdot \frac{1}{2\pi}$

$H(j\omega) =$



$\cdot \frac{1}{2\pi}$

$H(j\omega) =$



2) $x(t) = 2 + \cos(2t) + \sin(4t + \frac{\pi}{4})$

$x(t) = 2 + \cos(2t) + \sin(4(t + \pi/16))$

$\updownarrow \mathcal{F}$ Table 4.2 and Time Shift

$X(j\omega) = 4\pi\delta(\omega) + \pi\delta(\omega-2) + \pi\delta(\omega+2) + e^{j\omega\frac{\pi}{16}}\left[\frac{\pi}{j}\delta(\omega-4) - \frac{\pi}{j}\delta(\omega+4)\right]$

$Y(j\omega) = X(j\omega)H(j\omega)$

$Y(j\omega) = 4\pi H(j0)\delta(\omega) + \pi H(j2)\delta(\omega-2) + \pi H(-j2)\delta(\omega+2)$
$+ \frac{\pi}{j}e^{j(4)\frac{\pi}{16}}H(j4)\delta(\omega-4) - \frac{\pi}{j}e^{j(-4)\frac{\pi}{16}}H(-j4)\delta(\omega+4)$

For $H(j\omega)$ from part 1

$H(j0) = \pi \qquad H(j4) = 0$

$H(j2) = \pi/2 \qquad H(-j4) = 0$

$H(-j2) = \pi/2$

$\boxed{Y(j\omega) = 4\pi^2\delta(\omega) + \frac{\pi^2}{2}[\delta(\omega-2) + \delta(\omega+2)]}$

For $H(j\omega) = e^{-2|\omega|-1}$

$H(j0) = e^{-1} \qquad H(j2) = e^{-5} \qquad H(-j2) = e^{-5}$

$H(j4) = e^{-9} \qquad H(-j4) = e^{-9}$

$$\boxed{Y(j\omega) = 4\pi e^{-1}\delta(\omega) + \pi e^{-5}[\delta(\omega-2) + \delta(\omega+2)] + \frac{\pi}{j}e^{j\frac{\pi}{4}-9}\delta(\omega-4) - \frac{\pi}{j}e^{-(j\frac{\pi}{4}+9)}\delta(\omega+4)}$$

Question 4: [20%, Work-out question, Learning Objectives 3, 4, and 5] Consider a continuous time signal $x(t) = e^{-3t}\mathcal{U}(t)$. Let $y(t)$ denote the output if we pass $x(t)$ through an LTI system with impulse response $h(t) = e^{-(t-1)}\mathcal{U}(t-1)$. Find the expression of $y(t)$.

$$x(t) = e^{-3t}u(t) \qquad h(t) = e^{-(t-1)}u(t-1)$$

$\updownarrow \mathcal{F}$ Table 4.2 $\qquad \updownarrow \mathcal{F}$ Table 4.2 and Time shifting

$$X(j\omega) = \frac{1}{3+j\omega} \qquad \frac{e^{-j\omega}}{1+j\omega}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{e^{-j\omega}}{(1+j\omega)(3+j\omega)} = e^{-j\omega}\left(\frac{1}{(1+j\omega)(3+j\omega)}\right)$$

$$Y(j\omega) = e^{-j\omega}\left(\frac{A}{1+j\omega} + \frac{B}{3+j\omega}\right)$$

$$1 = A(3+j\omega) + B(1+j\omega)$$

$@\ j\omega = -1 \qquad A = \frac{1}{2}$

$@\ j\omega = -3 \qquad B = -\frac{1}{2}$

$$Y(j\omega) = \frac{.5e^{-j\omega}}{1+j\omega} - \frac{.5e^{-j\omega}}{3+j\omega}$$

$\updownarrow \mathcal{F}^{-1}$ Table 4.2 and Time shifting

$$\boxed{y(t) = .5e^{-(t-1)}u(t-1) - .5e^{-3(t-1)}u(t-1)}$$