Midterm #2 of ECE301, Section 2
8–9pm, Wednesday, March 4, 2015, EE 170.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solution Key

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [22%, Work-out question, Learning Objectives 1, 2, and 3]

1. [2%] What does the acronym "LTI" stand for?

Consider a discrete-time LTI system. We know that when the input is $x_1[n] = \mathcal{U}[n]$, the unit step signal, the output is

$$y_1[n] = \begin{cases} 0 & \text{if } n \leq -3 \\ 1 & \text{if } -2 \leq n \leq 1 \\ n & \text{if } 2 \leq n \leq 4 \\ 4 & \text{if } 5 \leq n \end{cases} \tag{1}$$

2. [10%] Find out the impulse response $h[n]$ of the system and plot $h[n]$ for the range of $n = -5$ to 5.

3. [10%] Consider another input signal

$$x_2[n] = e^{-jn} \tag{2}$$

Continuing from the previous question, find out the output $y_2[n]$ when the input is $x_2[n]$.

Hint 1: No need to plot $y_2[n]$.

Hint 2: If you do not know the answer to the previous question, you can assume that

$$h[n] = \begin{cases} 1 & \text{if } 0 \leq n \leq 2 \\ 2 & \text{if } 3 \leq n \leq 4 \end{cases}. \tag{3}$$

You will still get the full 12 points if your answer is correct.

1.) LTI means Linear Time Invariant

2.) if $X_1[n] = u[n] \longrightarrow \boxed{\text{sys}} \longrightarrow y_1[n]$

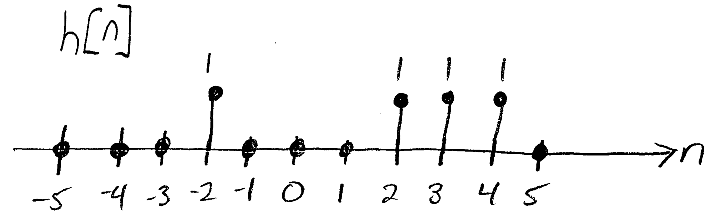
let $X_2[n] = X_1[n] - X_1[n-1] = u[n] - u[n-1] = \delta[n]$

because the system is LTI,

$X_2[n] = X_1[n] - X_1[n-1] \longrightarrow \boxed{\text{Sys}} \longrightarrow y_1[n] - y_1[n-1] = h[n]$

$$h[n] = y_1[n] - y_1[n-1] = \begin{cases} 0 & n \le -3 \\ 1 & -2 \le n \le 1 \\ n & 2 \le n \le 4 \\ 4 & 5 \le n \end{cases} - \begin{cases} 0 & n \le -2 \\ 1 & -1 \le n \le 2 \\ n-1 & 3 \le n \le 5 \\ 4 & 6 \le n \end{cases}$$

$$h[n] = \begin{cases} 0 & n \le -3 \\ 1 & n = -2 \\ 0 & -1 \le n \le 1 \\ 1 & 2 \le n \le 4 \\ 0 & 5 \le n \end{cases}$$



3.) $h[n] = \delta[n+2] + \delta[n-2] + \delta[n-3] + \delta[n-4]$

$X_2[n] = e^{-jn}$

$$y_2[n] = X_2[n] * h[n] = e^{-jn} * \delta[n+2] + e^{-jn} * \delta[n-2] + e^{-jn} * \delta[n-3] + e^{-jn} * \delta[n-4]$$

$$y_2[n] = e^{-j[n+2]} + e^{-j[n-2]} + e^{-j[n-3]} + e^{-j[n-4]}$$

$$\boxed{y_2[n] = e^{-j(n+2)} + e^{-j(n-2)} + e^{-j(n-3)} + e^{-j(n-4)}}$$

hint

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + 2\delta[n-3] + 2\delta[n-4]$$

$$X_2[n] * h[n] = e^{-jn} * (\delta[n] + \delta[n-1] + \delta[n-2] + 2\delta[n-3] + 2\delta[n-4])$$

$$= \boxed{e^{-jn} + e^{-j(n-1)} + e^{-j(n-2)} + 2e^{-j(n-3)} + 2e^{-j(n-4)}}$$

Question 2: [30%, Work-out question, Learning Objectives 1, 2, 3, and 5]

Consider a continuous time linear system for which the input/oupt relationship is

$$y(t) = \int_{-\infty}^{t} e^{-(t-s)} x(s) ds \tag{4}$$

1. [10%] Prove that the above system is also *time-invariant.*
2. [10%] Find the impulse response $h(t)$ of the above system.
3. [10%] Find the output when the input is $x(t) = 3e^{j100\pi t} + \cos(t)$.

   Hint: If you do not know the expression of $h(t)$ in the previous question, you can assume that $h(t) = e^{-|t-2|}$. You will get full 10 points if your answer is correct.

1.) system described by $x(t) \rightarrow \boxed{sys} \rightarrow \int_{-\infty}^{t} e^{-(t-s)} x(s)\, ds$

let $x(t-t_0) = x_1(t)$

$x_1(t) \xrightarrow{sys} \int_{-\infty}^{t} e^{-(t-s)} x_1(s)\, ds$

$= \int_{-\infty}^{t} e^{-(t-s)} x(s-t_0)\, ds$

let $t-s = u$, $ds = -du$

$t = s+u$

$s = t-u$

① $= \int_{-\infty}^{s+u} -e^{-u} x(t-u-t_0)\, du$

$y(t-t_0) = \int_{-\infty}^{t-t_0} e^{-(t-t_0-s)} x(s)\, ds$

let $t-t_0-s = u$, $ds = -du$

$t-t_0 = s+u$

$s = t-t_0-u$

② $= \int_{-\infty}^{s+u} -e^{-u} x(t-t_0-u)\, du$

since ① = ② , system is time-invariant

2.) to find the impulse response, let $x(t) = \delta(t)$

$\delta(t) \xrightarrow{sys} \int_{-\infty}^{t} e^{-(t-s)} \delta(s)\, ds = \int_{-\infty}^{t} e^{-(t-0)} \delta(s)\, ds = e^{-t} \int_{-\infty}^{t} \delta(s)\, ds$

$$\boxed{h(t) = \begin{cases} 0 & , t < 0 \\ e^{-t} & , t \geq 0 \end{cases}}$$

3.) $x(t) * h(t) = e^{-t}u(t) * \left[3e^{j100t} + \cos(t)\right]$

$$= e^{-t}u(t) * 3e^{j100t} + e^{-t}u(t) * \tfrac{1}{2}e^{jt} + e^{-t}u(t) * \tfrac{1}{2}e^{-jt}$$

recall

$$e^{j\omega t} \longrightarrow \boxed{LTI} \longrightarrow e^{j\omega t}\int_{-\infty}^{\infty} h(s)e^{-j\omega s}\,ds$$

$$= 3e^{j100t}\int_0^{\infty} e^{-s}e^{-j100s}\,ds + \tfrac{1}{2}e^{jt}\int_0^{\infty} e^{-s}e^{-js}\,ds + \tfrac{1}{2}e^{-jt}\int_0^{\infty} e^{-s}e^{js}\,ds$$

$$= \left.\frac{3e^{j100t}e^{-s(1+j100)}}{-(1+j100)} + \frac{\frac{1}{2}e^{jt}e^{-s(1+j)}}{-(1+j)} + \frac{\frac{1}{2}e^{-jt}e^{s(j-1)}}{(j-1)}\right]_0^{\infty}$$

$$= \boxed{\frac{3e^{j100t}}{1+j100} + \frac{e^{jt}}{2(1+j)} + \frac{e^{-jt}}{2(1-j)}}$$

alternate

$$h(t) = e^{-|t-2|} = \begin{cases} e^{2-t}, & t \geq 2 \\ e^{t-2}, & t < 2 \end{cases}$$

recall

$$e^{j\omega t} \longrightarrow \boxed{LTI} \longrightarrow e^{j\omega t}\int_{-\infty}^{\infty} h(s)e^{-j\omega s}\,ds$$

$$e^{j\omega t}\int_{-\infty}^{\infty} h(s)e^{-j\omega s}\,ds = e^{j\omega t}\left[\int_{-\infty}^{2} e^{s-2}e^{-j\omega s}\,ds + \int_{2}^{\infty} e^{2-s}e^{-j\omega s}\,ds\right]$$

$$= e^{j\omega t}\left[\int_{-\infty}^{2} e^{-2}e^{s(1-j\omega)}\,ds + \int_{2}^{\infty} e^{2}e^{-s(1+j\omega)}\,ds\right]$$

$$= e^{j\omega t}\left(\left[e^{-2}\frac{e^{s(1-j\omega)}}{1-j\omega}\right]_{\infty}^{2} + \left[e^{2}\frac{e^{-s(1+j\omega)}}{-(1+j\omega)}\right]_{2}^{\infty}\right.$$

$$= e^{j\omega t}\left[\frac{e^{-2+2-2j\omega}}{1-j\omega} + \frac{e^{2-2-2j\omega}}{1+j\omega}\right] = e^{j\omega t}\left[\frac{e^{-2j\omega}}{1-j\omega} + \frac{e^{-2j\omega}}{1+j\omega}\right]$$

$$= e^{j\omega(t-2)}\left[\frac{1+j\omega+1-j\omega}{(1+j\omega)(1-j\omega)}\right] = e^{j\omega(t-2)}\left[\frac{2}{1+\omega^2}\right]$$

---

now consider $x(t) = 3e^{j100\pi t} + \cos(t)$

$$= 3e^{j100\pi t} + \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\omega = 100\pi \qquad \omega = 1 \qquad \omega = -1$$

so,

$$= \frac{6e^{j100\pi(t-2)}}{1+(100\pi)^2} + \frac{e^{j(t-2)}}{2} + \frac{e^{-j(t-2)}}{2}$$

$$= \boxed{\frac{6e^{j100\pi(t-2)}}{1+(100\pi)^2} + \cos(t-2)}$$

Question 3: [14%, Work-out question, Learning Objectives 4, and 5]

Consider a continuous-time periodic signal

$$x(t) = \begin{cases} t^2 & \text{if } -1 \leq t \leq 1 \\ \text{periodic with period } T = 2 \end{cases}. \tag{5}$$

Obviously, the frequency is $\omega_0 = \frac{2\pi}{2}$. We denote the corresponding Fourier series coefficients by a_k, $k = 0, \pm 1, \pm 2, \cdots$.

1. [4%] Find the value of a_0.
2. [5%] Find the value of $\sum_{k=-\infty}^{\infty} a_k (-1)^k$.

Consider another continuous-time periodic signal

$$y(t) = \begin{cases} 2t & \text{if } -1 \leq t \leq 1 \\ \text{periodic with period } T = 2 \end{cases} \tag{6}$$

and we denote the corresponding Fourier series coefficients by b_k, $k = 0, \pm 1, \pm 2, \cdots$.

3. [3%] Express the values of b_k in terms of a_k.
4. [2%] Find the value of b_0.

1.) $a_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{1}{2}\int_{-1}^{1} t^2\,dt = \frac{1}{6}t^3\Big]_{-1}^{1} = \frac{1}{6} - \frac{-1}{6}$

$\boxed{a_0 = \frac{1}{3}}$

2.) Consider the FS representation of $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\pi t}$$

now, $\sum_{k=-\infty}^{\infty} a_k (-1)^k = \sum_{k=-\infty}^{\infty} a_k e^{j\pi k} = x(1) = 1^2 = 1$

$= \boxed{1}$

3.) $y(t) = \frac{d}{dt} x(t)$

$\Rightarrow$ differentiation property from table 3.1

$$x(t) \xrightarrow{FS} a_K$$

$$\frac{d\,x(t)}{dt} \xrightarrow{FS} jK \frac{2\pi}{T} a_K \quad , \; T = 2$$

$$\boxed{b_K = jK\pi a_K}$$

4.) $b_0 = \frac{1}{T} \int_T y(t)\,dt = \frac{1}{2} \int_{-1}^{1} 2t\,dt = \left. \frac{t^2}{2} \right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2}$

$$\boxed{b_0 = 0}$$

Question 4: [14%, Work-out question, Learning Objective 4] Consider a discrete time signal

$$x[n] = 2 + \cos\left(\frac{4\pi n}{3} + 1\right) + \sin\left(\frac{2\pi n}{5}\right). \tag{7}$$

Find the Fourier series representation of $x[n]$.

$$x[n] = 2 + \cos\left(\frac{4\pi}{3}\left(n + \frac{3}{4\pi}\right)\right) + \sin\left(\frac{2\pi}{5} n\right)$$

now consider $x[n] = x_1[n] + x_2\left[n + \frac{3}{4\pi}\right] + x_3[n]$ where

$x_1[n] = 2$

$x_2[n] = \cos\left(\frac{4\pi}{3} n\right)$ $\Rightarrow$ $x[n]$ is periodic with period

$x_3[n] = \sin\left(\frac{2\pi}{5} n\right)$ $N = \text{lcm}(1,3,5) = \underline{\underline{15}}$

now, if $x[n] \xrightarrow{FS} a_k$, $x_1[n] \xrightarrow{FS} b_k$, $x_2[n] \xrightarrow{FS} c_k$, $x_3[n] \xrightarrow{FS} d_k$ then

$$a_k = b_k + c_k e^{-jk\frac{2\pi}{N}\left(-\frac{3}{4\pi}\right)} + d_k$$

from linearity and time shifting properties

$x_1[n] = 2$ $\rightarrow$ periodic with period 15

$$b_k = \frac{1}{N} \sum_{k=\langle N \rangle} x_1[n] e^{-jk\frac{2\pi}{N} n}$$

$$b_k = \frac{1}{15} \sum_{k=0}^{14} 2\delta[n-k] e^{-jk\frac{2\pi}{15} n}$$

$$b_k = \frac{1}{15} \sum_{k=0}^{14} 2\delta[n-k] e^{-jk\frac{2\pi}{15}(n-k)}$$

1. $b_k = 2$

$$C_K = \frac{1}{N}\sum_{K=\langle N\rangle} X_2[n]e^{-jK\frac{2\pi}{N}n}$$

$$C_K = \frac{1}{15}\sum_{K=0}^{14}\cos\left(\frac{4\pi}{3}n\right)e^{-jK\frac{2\pi}{15}n} = \frac{1}{15}\sum_{K=0}^{14}\left(\frac{1}{2}e^{j\frac{4\pi n}{3}}+\frac{1}{2}e^{-j\frac{4\pi}{3}n}\right)e^{-j\frac{K2\pi}{15}n}$$

$$C_K = \frac{1}{30}\sum_{K=0}^{14} e^{j\frac{2\pi}{15}n(10-K)} + e^{-j\frac{2\pi}{15}n(10+K)}$$

if $K=\pm 10$, the summation equals 15, otherwise it is zero

also realize that $0 \le K \le 14$. Since -10 is not in this range and the signal is periodic with period 15 we have

$$C_K = \frac{1}{2}\delta[K-10] + \frac{1}{2}\delta[K-5]$$

now consider the time shift property

$$C_K e^{-jK\frac{2\pi}{N}n_0} = C_K e^{-jK\frac{2\pi}{15}\left(\frac{-3}{4\pi}\right)}$$

$$= \frac{1}{2}\delta[K-10]e^{\frac{-jK}{10}} + \frac{1}{2}\delta[K-5]e^{\frac{-jK}{10}}$$

$$\text{(2.)} = \frac{1}{2}e^{-j}\delta[K-10] + \frac{1}{2}e^{-j}\delta[K-5]$$

$$d_K = \frac{1}{N}\sum_{K=\langle N\rangle} X_3[n]e^{-jK\frac{2\pi}{N}n} = \frac{1}{15}\sum_{K=0}^{14}\frac{1}{2j}\left(e^{j\frac{2\pi n}{5}} - e^{-j\frac{2\pi n}{5}}\right)e^{-jK\frac{2\pi}{15}n}$$

$$= \frac{1}{30j}\sum_{K=0}^{14} e^{j\frac{2\pi}{15}n(3-K)} - e^{-j\frac{2\pi}{15}n(3+K)}$$

if $K=3$, sum $=15$; if $K=-3$ or $K=-3+15=12$, sum $=-15$; otherwise sum $=0$

same reason as above

$$\text{(3)}\ d_K = \frac{1}{2j}\delta[K-3] - \frac{1}{2j}\delta[K-12]$$

So, the answer is (1) + (2) + (3)

$$\boxed{a_K = 2 + \frac{1}{2}e^{-j}\delta[K-10] + \frac{1}{2}e^{-j}\delta[K-5] + \frac{1}{2j}\delta[K-3] - \frac{1}{2j}\delta[K-12]}$$

Question 5: [20%, Multiple Choices]

The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \begin{cases} \int_{t-1}^{t} x_1(s)ds & \text{if } x_1(t+1) \geq 0 \\ \int_{t-2}^{t} x_1(s)ds & \text{if } x_1(t+1) < 0 \end{cases} \tag{8}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = e^{j(x_2[n]-x_2[-n])} \cdot x_2[n] \tag{9}$$

Answer the following questions

1. [4%, Learning Objective 1] Is System 1 memoryless? Is System 2 memoryless?
2. [4%, Learning Objective 1] Is System 1 causal? Is System 2 causal?
3. [4%, Learning Objective 1] Is System 1 stable? Is System 2 stable?
4. [4%, Learning Objective 1] Is System 1 linear? Is System 2 linear?
5. [4%, Learning Objective 1] Is System 1 time-invariant? Is System 2 time-invariant?

1.) System 1 has memory
System 2 has memory

2.) System 1 is not causal
System 2 is not causal

3.) system 1 is stable
system 2 is stable

4.) System 1 is not linear
System 2 is not linear

5.) system 1 is time invariant
system 2 is not time-invariant

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t - \tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

three examples, we illustrate this. The last example in this section then demonstrates how properties of a signal can be used to characterize the signal in great detail.

Example 3.6

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 3.10. We could determine the Fourier series representation of $g(t)$ directly from the analysis equation (3.39). Instead, we will use the relationship of $g(t)$ to the symmetric periodic square wave $x(t)$ in Example 3.5. Referring to that example, we see that, with $T = 4$ and $T_1 = 1$,

$$g(t) = x(t - 1) - 1/2.$$

Discrete-time Fourier series

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \tag{1}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \tag{2}$$

Continuous-time Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \tag{3}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \tag{4}$$

Continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \tag{5}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{6}$$

Discrete-time Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(j\omega) e^{j\omega n} d\omega \tag{7}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{8}$$

Laplace transform

$$x(t) = \frac{1}{2\pi} e^{\sigma t} \int_{-\infty}^{\infty} X(\sigma + j\omega) e^{j\omega t} d\omega \tag{9}$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{10}$$

Z transform

$$x[n] = r^n \mathcal{F}^{-1}(X(re^{j\omega})) \tag{11}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{12}$$