Midterm #1 of ECE301, Section 2

8–9pm, Wednesday, February 4, 2015, EE 170.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name: Solution Key

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [20%, Work-out question, Learning Objective 3] Consider two discrete-time signals $x[n]$ and $y[n]$.

$$x[n] = \begin{cases} n2^{n-1} & \text{if } 1 \le n \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = \begin{cases} e^{n} & \text{if } n \le 10 \\ 0 & \text{otherwise} \end{cases}$$

Compute the expression of

$$z[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k].$$

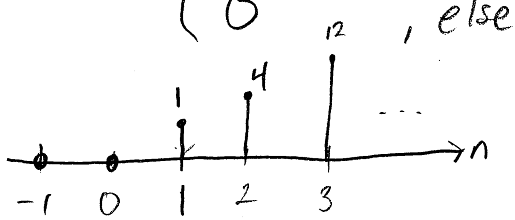
Hint 1: You may need to use the following two formulas: Suppose $|r| < 1$. Then we have

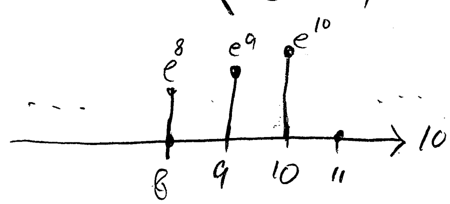
$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} akr^{k-1} = \frac{a}{(1-r)^2}. \tag{2}$$

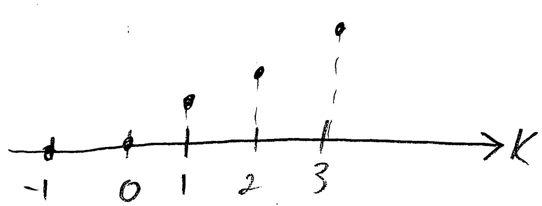
Hint 2: You need to study different cases in your answer. The computation of some cases may be significantly longer and you may want to work on some other questions first before you come back to finish the computation of some of the cases.

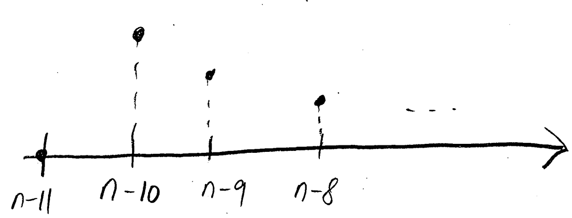
$$x[n] = \begin{cases} n2^{n-1} & , n \ge 1 \\ 0 & , \text{else} \end{cases} \qquad y[n] = \begin{cases} e^{n} & , n \le 10 \\ 0 & , \text{else} \end{cases}$$



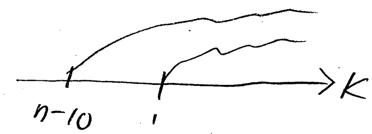


$$x[k] = \begin{cases} k2^{k-1} & , k \ge 1 \\ 0 & , \text{else} \end{cases} \qquad y[n-k] = \begin{cases} e^{n-k} & , k \ge n-10 \\ 0 & , \text{else} \end{cases}$$





Case 1 $\quad n-10<1 \Rightarrow n<11$

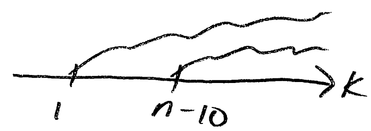


$$\sum_{k=1}^{\infty} x[k]\,y[n-k] = \sum_{k=1}^{\infty} k\,2^{(k-1)}\,e^{n-k}$$

$$= e^{n}\sum_{k=1}^{\infty} k\,2^{k-1}e^{-k} = e^{n}\sum_{k=1}^{\infty} k\,2^{k-1}\left(\frac{1}{e}\right)^{k-1}\left(\frac{1}{e}\right)$$

$$= e^{n-1}\sum_{k=1}^{\infty} k\left(\frac{2}{e}\right)^{k-1} = \frac{e^{n-1}}{\left(1-\frac{2}{e}\right)^{2}}$$

Case 2 $\quad n-10\geq 1 \Rightarrow n\geq 11$



$$\sum_{k=n-10}^{\infty} x[k]\,y[n-k] = \sum_{k=n-10}^{\infty} k\,2^{k-1}e^{n-k}$$

let $l = k-n+11 \Rightarrow k = l+n-11$

$$= \sum_{l=1}^{\infty}(l+n-11)\,2^{l+n-12}\,e^{n-(l+n-11)} = \sum_{l=1}^{\infty} l\,2^{l+n-12}e^{-l+11} + (n-11)\,2^{l+n-12}e^{-l+11}$$

$$= \sum_{l=1}^{\infty} l\,2^{l-1}2^{n-11}\left(\frac{1}{e}\right)^{l-1}e^{11}\cdot\left(\frac{1}{e}\right) + (n-11)\sum_{l=1}^{\infty}2^{l-1}2^{n-11}e^{11}\left(\frac{1}{e}\right)^{l-1}\cdot\left(\frac{1}{e}\right)$$

$$= 2^{n-11}e^{10}\sum_{l=1}^{\infty} l\left(\frac{2}{e}\right)^{l-1} + (n-11)\,e^{10}\,2^{n-11}\sum_{l=1}^{\infty}\left(\frac{2}{e}\right)^{l-1}$$

$$= \frac{2^{n-11}e^{10}}{\left(1-\frac{2}{e}\right)^{2}} + \frac{(n-11)\,e^{10}\,2^{n-11}}{1-\frac{2}{e}}$$

$$\boxed{z[n] = \begin{cases} \dfrac{e^{n-1}}{\left(1-\frac{2}{e}\right)^{2}}, & n<11 \\[2ex] \dfrac{2^{n-11}e^{10}}{\left(1-\frac{2}{e}\right)^{2}} + \dfrac{(n-11)e^{10}2^{n-11}}{1-\frac{2}{e}}, & n\geq 11 \end{cases}}$$

Question 2: [16%, Work-out question, Learning Objectives 1, and 4] Given a continuous-time signal

$$x(t) = \begin{cases} a^{-t} & \text{if } 2 \leq t \\ 0 & \text{otherwise} \end{cases}$$

where a is a constant real-valued parameter satisfying $a > 0$, the value of which will be given later.

1. [3%] Assuming $a = 0.5$, plot $x(t)$ for the range of $t = -5$ to 5.
2. [5%] Assuming $a = 0.5$, compute the average power $x(t)$ for the duration of $t = -5$ to 5.
3. [5%] Assuming $a = 2$, compute the total energy of $x(t)$.
4. [3%] What is the range of a (assuming $a > 0$) for which $x(t)$ is of finite energy?

1.) $x(t) = \begin{cases} (.5)^{-t} & , 2 \leq t \\ 0 & , \text{else} \end{cases}$

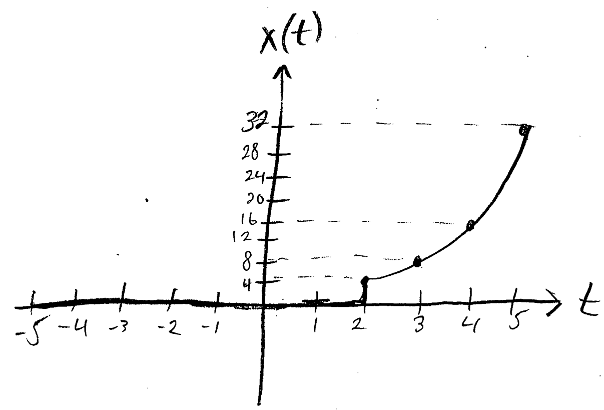
if $-5 \leq t < 2$, $x(t) = 0$

if $t = 2 \Rightarrow x(t) = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4$

if $t = 3 \Rightarrow x(t) = \left(\frac{1}{2}\right)^{-3} = 2^3 = 8$

if $t = 4 \Rightarrow x(t) = \left(\frac{1}{2}\right)^{-4} = 2^4 = 16$

if $t = 5 \Rightarrow x(t) = \left(\frac{1}{2}\right)^{-5} = 2^5 = 32$



2.) Average power

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt = \frac{1}{5-(-5)} \int_{2}^{5} (2^t)^2 dt = \frac{1}{10} \int_{2}^{5} 2^{2t} dt$$

$$= \frac{1}{10} \int_{2}^{5} 4^t dt = \frac{1}{10} \left. \frac{4^t}{\ln(4)} \right]_2^5 = \frac{1}{10} \left(\frac{4^5}{\ln 4} - \frac{4^2}{\ln(4)} \right)$$

$$= \boxed{\frac{1008}{10 \ln(4)}}$$

3.) $$x(t) = \begin{cases} 2^{-t}, & t \geq 2 \\ 0, & \text{else} \end{cases}$$

$$\text{Total energy} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{2}^{\infty} \left(2^{-t}\right)^2 dt = \int_{2}^{\infty} \left(\tfrac{1}{2}\right)^{2t} dt$$

$$= \int_{2}^{\infty} \left(\tfrac{1}{4}\right)^t dt = \left.\frac{\left(\frac{1}{4}\right)^t}{\ln\left(\frac{1}{4}\right)}\right]_2^{\infty} = \left(0 - \frac{\left(\frac{1}{4}\right)^2}{\ln\left(\frac{1}{4}\right)}\right)$$

$$= \frac{\frac{1}{16}}{\ln(4)} = \boxed{\frac{1}{16\ln(4)}}$$

4.) If $0 < a \leq 1 \Rightarrow x(t)$ has infinite energy

so $\boxed{a > 1 \Rightarrow x(t) \text{ has finite energy}}$

notice $X(t)$ is a decreasing exponential if $a > 1$, a constant if $a = 1$, and an exponential if $a < 1$ for values of $t \geq 2$.

Question 3: [14%, Work-out question, Learning Objectives 1, 2 and 4] Consider the following function.

$$X(z) = \frac{1}{(z - \frac{1}{3})(2z-1)}.$$

1. [6%] Find the value of $|X(1+j)|^2$. Hint: Basically we set $z = (1+j)$ and then evaluate the value of $|X(z)|^2$.

2. [8%] Express $X(z)$ in the following form:

$$X(z) = \frac{a}{z+b} + \frac{c}{z+d}. \tag{3}$$

That is, you are asked to find the a, b, c, and d values.

1.) $X(z) = \dfrac{1}{2z^2 - z - \frac{2}{3}z + \frac{1}{3}} = \dfrac{1}{2z^2 - \frac{5}{3}z + \frac{1}{3}}$

$X(1+j) = \dfrac{1}{2(1+j)^2 - \frac{5}{3}(1+j) + \frac{1}{3}} = \dfrac{1}{2(1+2j+j^2) - \frac{5}{3} - \frac{5}{3}j + \frac{1}{3}} = \dfrac{1}{2+4j-2-\frac{4}{3}-\frac{5}{3}j}$

$X(1+j) = \dfrac{1}{\frac{7}{3}j - \frac{4}{3}}$

$|X(1+j)|^2 = \left(\dfrac{1}{\sqrt{\left(\frac{7}{3}\right)^2 + \left(\frac{4}{3}\right)^2}}\right)^2 = \dfrac{1}{\frac{49}{9} + \frac{16}{9}} = \boxed{\dfrac{9}{65}}$

2.) $X(z) = \dfrac{1}{(z-\frac{1}{3})(2z-1)} = \dfrac{p}{z-\frac{1}{3}} + \dfrac{q}{2z-1}$

$1 = p(2z-1) + q(z-\frac{1}{3})$

$1 = 2pz - p + qz - q\frac{1}{3}$

$$\begin{cases} 2p + q = 0 \\ -p - \frac{1}{3}q = 1 \end{cases}$$

$$\begin{cases} 2p + q = 0 \\ -p - \frac{1}{3}q = 1 \end{cases}$$

$$p = -\frac{1}{3}q - 1$$

$$2\left(-\frac{1}{3}q - 1\right) + q = 0$$

$$-\frac{2}{3}q - 2 + q = 0$$

$$\frac{1}{3}q = 2$$

$$\underline{q = 6}$$

$$p = -\frac{1}{3}(6) - 1 = -2 - 1 = -3$$

$$X(z) = \frac{-3}{z - \frac{1}{3}} + \frac{6}{2z - 1} = \frac{a}{z + b} + \frac{c}{z + d}$$

$$\boxed{\begin{aligned} a &= -3 \\ b &= -1/3 \\ c &= 3 \\ d &= -1/2 \end{aligned}}$$

Question 4: [16%, Work-out question, Learning Objective 1] Consider the following signal.

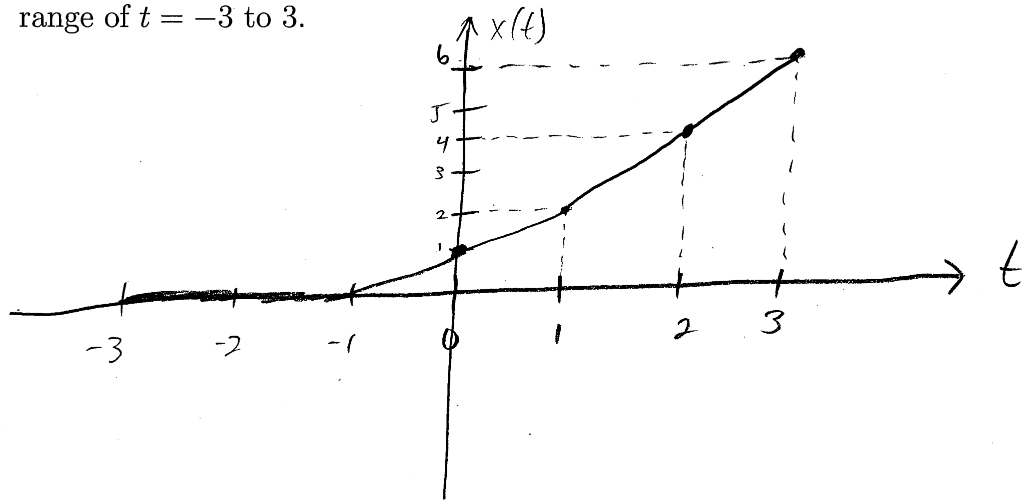
$$x(t) = (t+1)U(t+1) + (t-1)U(t-1) \tag{4}$$

1. [8%] Plot $x(t)$ for the range of $t = -3$ to 3.
2. [8%] Plot the even part of $x(t)$ for the range of $t = -3$ to 3.

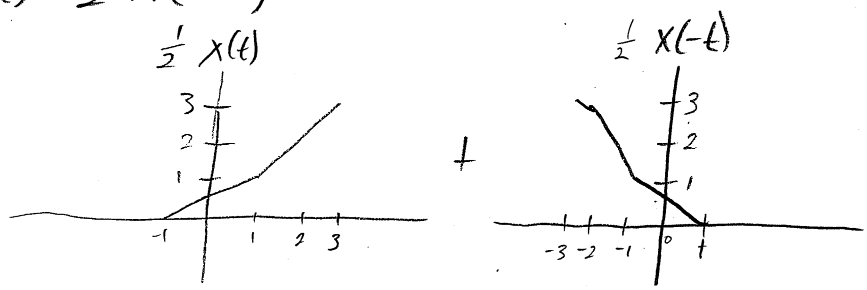
   Hint: If you do not know how to plot $x(t)$, you can solve the following alternative question instead. You will still get 6 points if your answer is correct.

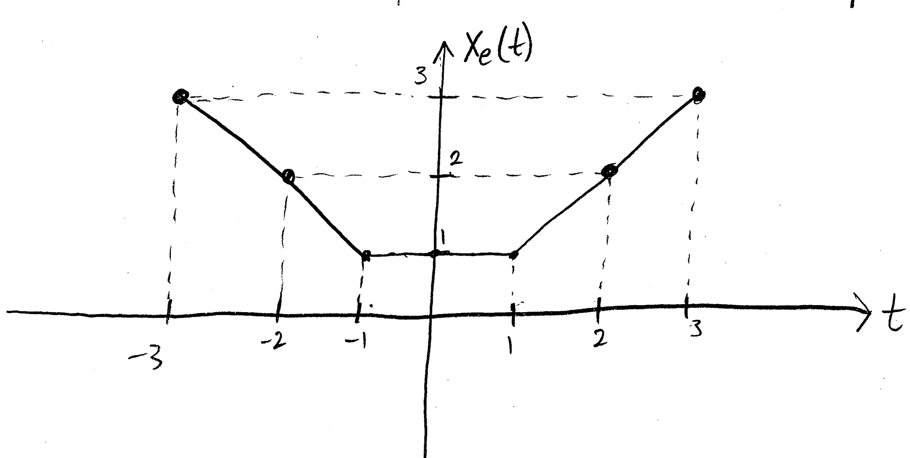
   Alternative question: Suppose $y(t) = (t+2.5)^2$. Plot the odd part of $y(t)$ for the range of $t = -3$ to 3.

1.)



2.) $X_e(t) = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$





Alternative Question

$$y(t) = (t+2.5)^2 \qquad y_o(t) = \tfrac{1}{2}y(t) - \tfrac{1}{2}y(-t)$$

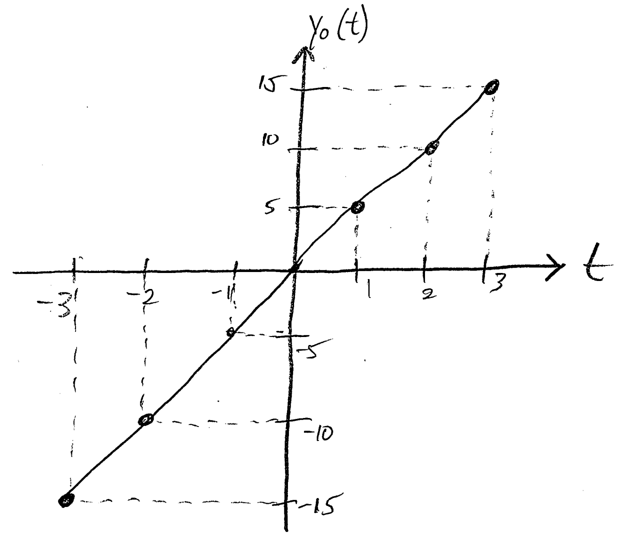
$$y(t) = t^2 + 5t + 6.25$$

$$y(-t) = t^2 - 5t + 6.25$$

$$y(t) - y(-t) = t^2 + 5t + 6.25 - t^2 + 5t - 6.25 = 10t$$

$$y_o(t) = \tfrac{1}{2}y(t) - \tfrac{1}{2}y(-t) = \tfrac{1}{2}\left(y(t) - y(-t)\right) = \underline{\underline{5t}}$$

plot over $-3 \leq t \leq 3$



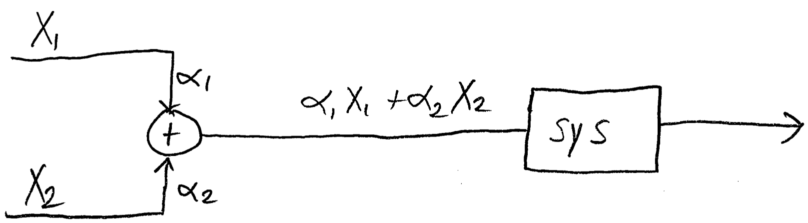
Question 5: [14%, Work-out question, Learning Objective 1]

Consider the following system that takes signal $x(t)$ as input and outputs

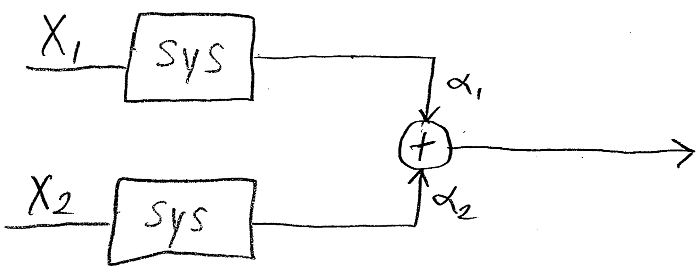
$$y(t) = t^2 \cdot x(t+1) \qquad (5)$$

Is the above system is linear or not? Carefully explain the steps how you decide whether the system is linear or not.

config 1



config 2



are they equal?

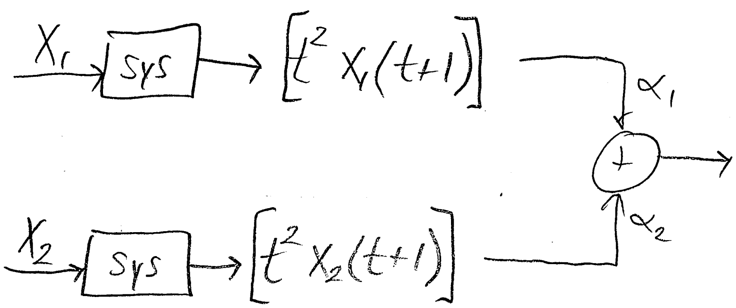
yes ⇒ linear

no ⇒ not linear

let $X_3 = \alpha_1 X_1 + \alpha_2 X_2$

$$X_3 \to \boxed{Sys} \to t^2 x_3(t+1) = t^2\left(\alpha_1 X_1(t+1) + \alpha_2 X_2(t+1)\right)$$

$$= \alpha_1 t^2 X_1(t+1) + \alpha_2 t^2 X_2(t+1)$$



$$= \alpha_1 t^2 x_1(t+1) + \alpha_2 t^2 x_2(t+1)$$

⇒ Since the output of the two configurations are the same, the system $\boxed{\text{is Linear}}$

Question 6: [20%, Multiple Choices] The following questions are multiple-choice questions and there is no need to justify your answers. Consider two continuous-time signals:

$$x_1(t) = e^{2\sin(t)} + e^{-2\sin(t)}$$
$$x_2(t) = \frac{\cos(3t) + \sin(2\pi t)}{\cos(2t)}$$

and two discrete-time signals:

$$x_3[n] = \cos(n^3) + \sin(\frac{n}{3})$$
$$x_4[n] = n^2U[n-1] - n^2U[n+1].$$

1. [10%, Learning Objective 1] For $x_1(t)$ to $x_4[n]$, determine whether it is periodic or not, *respectively*. If it is periodic, write down the fundamental period. Please state explicitly which signal is periodic and which is not.

2. [10%, Learning Objective 1] For $x_1(t)$ to $x_4[n]$, determine whether it is even or odd or neither of them, respectively. Please state explicitly which signal is even, which is odd, and which is neither.

1.) $X_1(t)$ is periodic with fundamental period π

$X_2(t)$ is not periodic

$X_3[n]$ is not periodic

$X_4[n]$ is not periodic

2.) $X_1(t)$ is even

$X_2(t)$ is neither

$X_3[n]$ is neither

$X_4[n]$ is even

Correction: x_4[n] is neither even nor odd.