Midterm #3 of ECE301, Section 2
8–9pm, Monday, November 9, 2015, ARMS 1010.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**

2. This is a closed book exam.

3. This exam contains multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.

4. Use the back of each page for rough work.

5. Neither calculators nor help sheets are allowed.

Name: SOLUTIONS

Student ID:

I certify that I have neither given nor received unauthorized aid on this exam.

Signature: Date:

Question 1: [18%, Work-out question, Learning Objectives 1, 2, 3] Consider a discrete-time signal

$$x[n] = \begin{cases} 0.5e^{jn} & \text{if } 1 \le n \le 10 \\ 0.5 & \text{if } 11 \le n \le 20 \text{ .} \\ \text{is periodic with period 20} \end{cases} \tag{1}$$

Denote the Fourier series coefficient of $x[n]$ by (a_k, ω_0).

1. [6%] Find the value of a_0.

2. [6%] Find the value of $\sum_{k=0}^{19} a_k$.

3. [6%] Find the value of $\sum_{k=0}^{19} |a_k|^2$.

Hint: You may need to use the following formulas: If $|r| \ne 1$, then we have

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a(1 - r^K)}{1 - r}. \tag{2}$$

If $|r| < 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1 - r}. \tag{3}$$

If $|r| \ge 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} \text{ does not exist.} \tag{4}$$

1. $\quad \omega_0 = \dfrac{2\pi}{N} = \dfrac{2\pi}{20}$

$$a_0 = \frac{1}{20} \sum_{n=1}^{20} x[n] = \frac{1}{20}\left[\sum_{n=1}^{10} 0.5\, e^{jn} + \sum_{n=11}^{20} 0.5 \right]$$

$$= \frac{1}{20}\left[e^{j} \sum_{n=1}^{10} 0.5\, e^{j(n-1)} + 0.5 \times 10 \right]$$

$$= \frac{1}{20}\left[e^{j} \times \frac{0.5(1 - e^{j10})}{1 - e^{j}} + 5 \right]$$

$$a_0 = \frac{0.5\, e^{j}(1 - e^{j10})}{20(1 - e^{j})} + \frac{1}{4}$$

2. $x[n] = \sum_{k=0}^{19} a_k \, e^{jkw_0 n}$

$x[0] = \sum_{k=0}^{19} a_k \, e^{jkw_0 \times 0}$

$\Rightarrow \sum_{k=0}^{19} a_k = x[0] = x[0+N] = x[0+20]$

$= x[20]$

$= 0.5$

3. From Parseval's Relation

$\sum_{k=0}^{19} |a_k|^2 = \dfrac{1}{20} \sum_{n=1}^{20} |x[n]|^2$

$= \dfrac{1}{20} \left[\sum_{n=1}^{10} |0.5 \, e^{jn}|^2 + \sum_{n=11}^{20} |0.5|^2 \right]$

$|0.5 \, e^{jn}|^2 = |0.5|^2 |e^{jn}|^2 = |0.5|^2 = 0.25$

$\Rightarrow \sum_{k=0}^{19} |a_k|^2 = \dfrac{1}{20} \sum_{n=1}^{20} 0.25 = \dfrac{1}{20} \times 0.25 \times 20$

$= 0.25$

Question 2: [23%, Work-out question, Learning Objectives 2, 3, and 4] Consider a CT-LTI system, for which the impulse response is

$$h(t) = \frac{\sin(5t) - \sin(3t)}{\pi t} \tag{5}$$

Consider a CT signal:

$$x(t) = \sum_{k=0}^{\infty} \sin(k\pi/2t) \cdot \sin\left(\frac{k\pi t}{2}\right) \tag{6}$$

1. [10%] Find the Fourier series $(a_k, \frac{2\pi}{N})$ of $x(t)$. Specifically, (i) find the expression of a_k for any arbitrary k, (ii) find the period N, and (iii) plot a_k for the range of $-2 \le k \le 2$.

2. [13%] Let $y(t)$ denote the output when the input is $x(t)$. Find the Fourier series $(b_k, \frac{2\pi}{N})$ of $y(t)$. Specifically, (i) find the expression of b_k, (ii) find the period N, and (iii) plot b_k for the range of $-3 \le k \le 3$.

   Hint 1: If you do not know the answer to this sub-question, you can find the Fourier transform $H(j\omega)$ of $h(t)$. You will receive 10 points if your answer is correct.

   Hint 2: The following values may be of use.

$$\frac{\pi}{2} \approx 1.571 \tag{7}$$

$$\pi \approx 3.142 \tag{8}$$

$$\frac{3\pi}{2} \approx 4.712 \tag{9}$$

$$2\pi \approx 6.283 \tag{10}$$

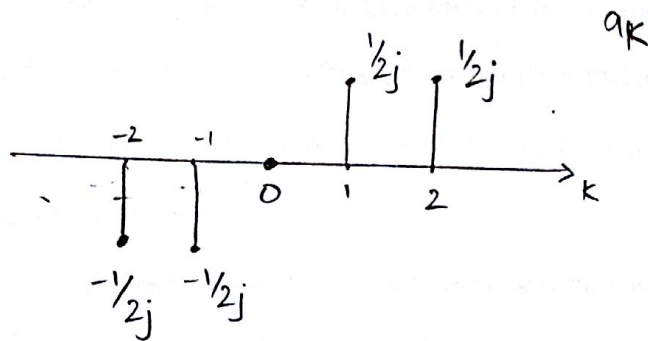$$\frac{5\pi}{2} \approx 7.854 \tag{11}$$

1. period of each sinewave $= \dfrac{2\pi}{k\pi/2} = \dfrac{4}{k}$

Common period for sum of all sinewaves $= \text{LCM}\left(\dfrac{4}{1}, \dfrac{4}{2}, \dfrac{4}{3}, \cdots, \dfrac{4}{k}, \cdots\right)$
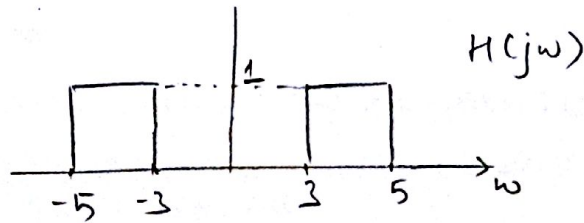
$$N = 4$$

$$\Rightarrow \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

each term in sum $= \sin\left(k\frac{\pi}{2}t\right) = \dfrac{e^{jk\omega_0 t} - e^{-jk\omega_0 t}}{2j}$

$$= \frac{1}{2j} e^{jk\omega_0 t} + \left(\frac{-1}{2j}\right) e^{j(-k)\omega_0 t}$$

By observation, $a_K = \dfrac{\text{sgn}(k)}{2j}$ $\forall k \in \mathbb{Z}$

a_K

2. $h(t) = \dfrac{\sin 5t}{\pi t} - \dfrac{\sin 3t}{\pi t}$ $\xleftrightarrow{F.T.}$ $H(j\omega) = (u(\omega+5) - u(\omega-5))$
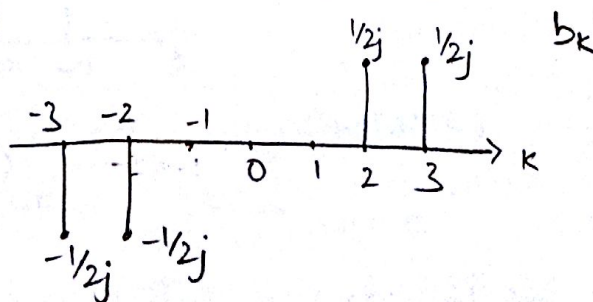$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (u(\omega+3) - u(\omega-3))$

$H(j\omega)$

in $x(t)$

each term $\sin \dfrac{k\pi}{2} t$ has two spikes in frequency at $\omega = \pm \dfrac{k\pi}{2}$

Due to the nature of $H(j\omega)$, $Y(j\omega)$ contains only those spikes lying in $3 < |\omega| < 5$. Only π & $\dfrac{3\pi}{2}$ lie in this range, so $y(t)$ consists of only those sine terms corresponding to $k = 2, 3$.

$\Rightarrow y(t) = \sin \pi t + \sin \dfrac{3\pi}{2} t$; $T = LCM\left(\dfrac{2\pi}{\pi}, \dfrac{2\pi}{\frac{3\pi}{2}}\right)$

$= \dfrac{e^{j\pi t} - e^{-j\pi t}}{2j} + \dfrac{e^{j\frac{3\pi}{2}t} - e^{-j\frac{3\pi}{2}t}}{2j}$ $= LCM\left(2, \frac{4}{3}\right) = \underline{\underline{4 = N}}$

By observation, $b_K = \dfrac{\text{sgn}(k)}{2j}$ for $k = 2, 3$

b_K

Question 3: [25%, Work-out question, Learning Objectives 4, and 5]

1. [9%] Consider a continuous time signal $x(t)$ with its Fourier transform $X(j\omega)$ being

$$X(j\omega) = (1+j)\delta(\omega - 3) + (1-j)\delta(\omega + 3) + \delta(\omega) \qquad (12)$$

Find the expression of $x(t)$.

2. [3%] Is $x(t)$ a real-valued signal? Use one quick sentence to justify your answer.

3. [1%] Consider a continuous time signal $y(t)$:

$$y(t) = \sum_{k=-2}^{2} \delta(t - 2\pi k) \qquad (13)$$

Is $y(t)$ periodic?
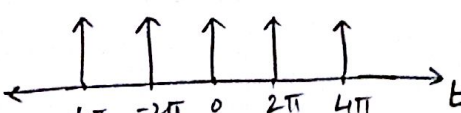
4. [10%] Find the expression of its Fourier transform $Y(j\omega)$.

5. [2%] Is $Y(j\omega)$ periodic? If so, what is its period? If not, use a quick sentence to justify why $Y(j\omega)$ is not periodic.

1. $X(j\omega) = \dfrac{\pi}{\pi} \left[\delta(\omega-3) + \delta(\omega+3) \right] + \dfrac{\pi}{\pi}\dfrac{(-1)}{j} \left[\delta(\omega-3) - \delta(\omega+3) \right]$

   $+ \dfrac{2\pi}{2\pi} \delta(\omega)$

   From the table of FT pairs

   $x(t) = \dfrac{\cos 3t}{\pi} - \dfrac{\sin 3t}{\pi} + \dfrac{1}{2\pi}$

2. Yes, since $|X(j\omega)|$ is symmetric

   Alternately since it is a "combination of real signals

3.

   $y(t)$ is not periodic.

4. $\delta(t - 2\pi k) \xrightarrow{F.T.} e^{-j\omega(2\pi k)}$

   $Y(j\omega) = e^{-j4\pi\omega} + e^{j4\pi\omega} + e^{-j2\pi\omega} + e^{j2\pi\omega} + e^{j0}$

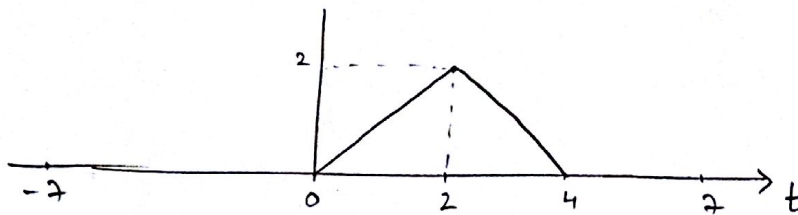 $= 2\cos(4\pi\omega) + 2\cos(2\pi\omega) + 1$

5. $Y(j\omega)$ is periodic

   Period $= LCM\left(\dfrac{2\pi}{4\pi}, \dfrac{2\pi}{2\pi}\right) = LCM\left(\dfrac{1}{2}, 1\right) = \underline{\underline{1}}$ radian

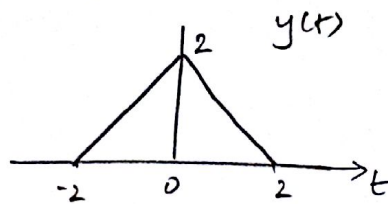*Question 4:* [15%, Work-out question, Learning Objectives 4, 5, and 6] Consider the following signal

$$x(t) = \begin{cases} t & \text{if } 0 \le t < 2 \\ 4-t & \text{if } 2 \le t < 4 \\ 0 & \text{otherwise.} \end{cases} \qquad (14)$$

1. [2%] Plot $x(t)$ for the range of $-7 < t < 7$.

2. [13%] Find the Fourier transform $X(j\omega)$ of $x(t)$.

1.

2. considir   y(t) below   considir z(t) below
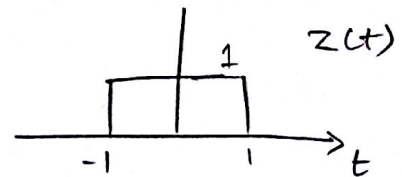
$$y(t) = z(t) * z(t)$$

$$\Rightarrow \quad Y(j\omega) = Z(j\omega) \cdot Z(j\omega) = Z(j\omega)^2$$

$$= \left(\frac{2 \sin \omega}{\omega} \right)^2 \cdot \frac{4 \sin^2 \omega}{\omega^2}$$

Now $x(t) = y(t-2) \xleftrightarrow{F.T.} X(j\omega) = e^{-j\omega 2} Y(j\omega)$

$$\therefore \quad X(j\omega) = \frac{4 e^{-j\omega 2} \sin^2 \omega}{\omega^2}$$

Question 5: [19%, Multiple Choices] Consider the following AM transmission system. For the transmitter side, for whatever signal $x(t)$, we first pass it through a low-pass filter with cut-off frequency 0.5Hz. Then we multiply it with $\cos(100\pi t)$.

For the receiver side, we first multiply the received signal by $\cos(100\pi t)$ and then pass it through a low-pass filter with cut-off frequency 1Hz. The overall system flow chart is as follows.
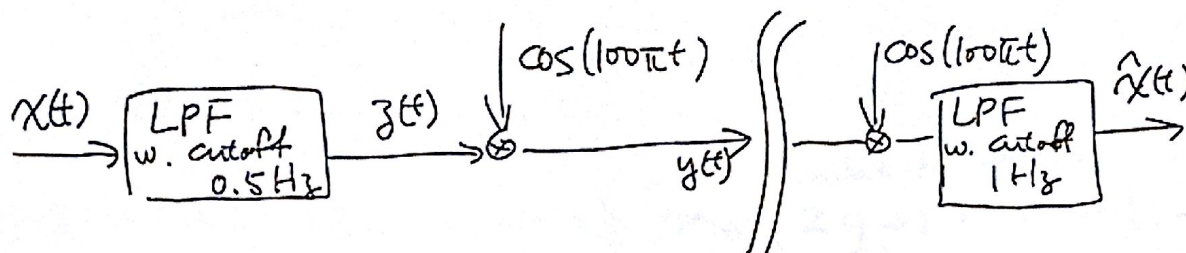
Answer the following questions:

1. [3%] What is the expression of the impulse response of a low-pass filter with cutoff frequency 0.5Hz.

2. [3%] If the original signal $x(t)$ has $X(j\omega) = U(\omega + 5) - U(\omega - 5)$. What is the Fourier transform $Z(j\omega)$ of $z(t)$?

   Hint: If you do not know the answer to the previous subquestion, you can assume that a low-pass filter with cutoff frequency 0.5Hz has $H(j\omega)$ being $U(\omega+3)-U(\omega-3)$. You will get full credit if your $Z(j\omega)$ is correct.

3. [9%] Plot the signal $y(t)$ for the range of $-4 < t < 4$.

   Hint 1: If you do not know how to plot $y(t)$, you can simply plot $z(t)$ instead. You will receive 5 points if your plot of $z(t)$ is correct.

   Hint 2: If you do not know the answers of the previous questions, you can answer the following question instead: For a frequency-domain signal $X_1(j\omega) = U(\omega+3) - U(\omega - 3)$, plot assume $x_1(t)$ for the range of $-\pi < t < \pi$. You will receive 5 points if your answer is correct.
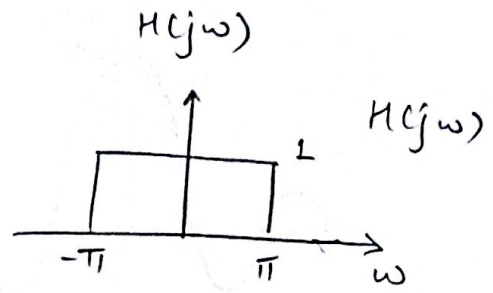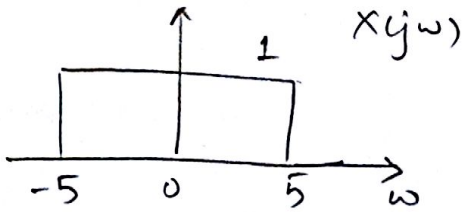
4. [4%, Learning Objective 1] If you play the signals $z(t)$ and $\hat{x}(t)$ using the speakers of your computer. Will these two signals sound identical? Use a couple of sentences to justify your answer.

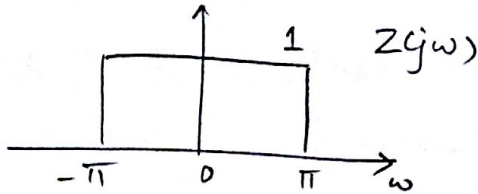1. $H(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases}$

$\omega_c = 2\pi f_c$

$\Rightarrow \omega_c = 2\pi \times (0.5 \, Hz)$

$\omega_c = \pi \, rad$

$\Rightarrow h(t) = \dfrac{\sin \pi t}{\pi t}$
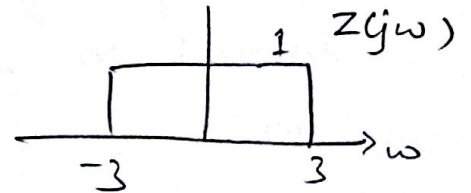
2. $X(j\omega) = u(\omega+5) - u(\omega-5)$ $H(j\omega)$
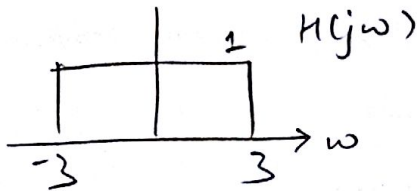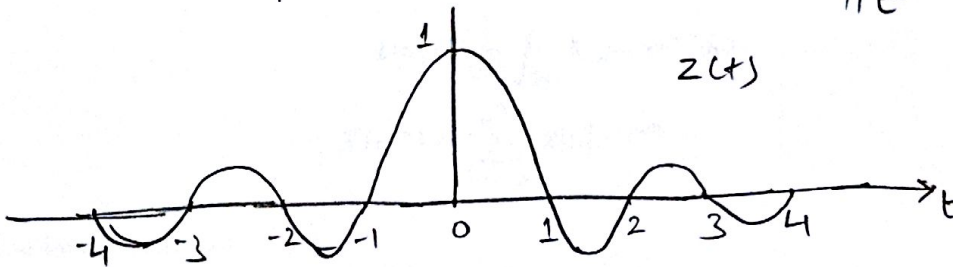

$X(j\omega)$

$H(j\omega)$

$Z(j\omega) = X(j\omega) H(j\omega) = u(\omega+\pi) - u(\omega-\pi)$

$Z(j\omega)$

if $H(j\omega) = u(\omega+3) - u(\omega-3)$, then, $Z(j\omega) = u(\omega+3) - u(\omega-3)$
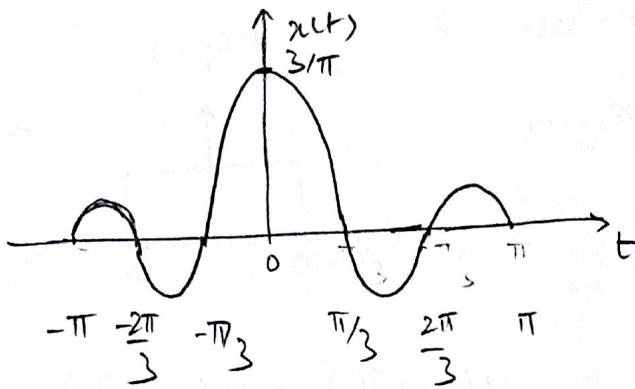
similar to above

$H(j\omega)$

$Z(j\omega)$

3.   From the FT pair tables, $Z(t) = \dfrac{\sin \pi t}{\pi t}$

$Z(t)$

$y(t) = Z(t) \cos(100\pi t) = \dfrac{\sin \pi t}{\pi t} \cos(100\pi t)$

$y(t)$
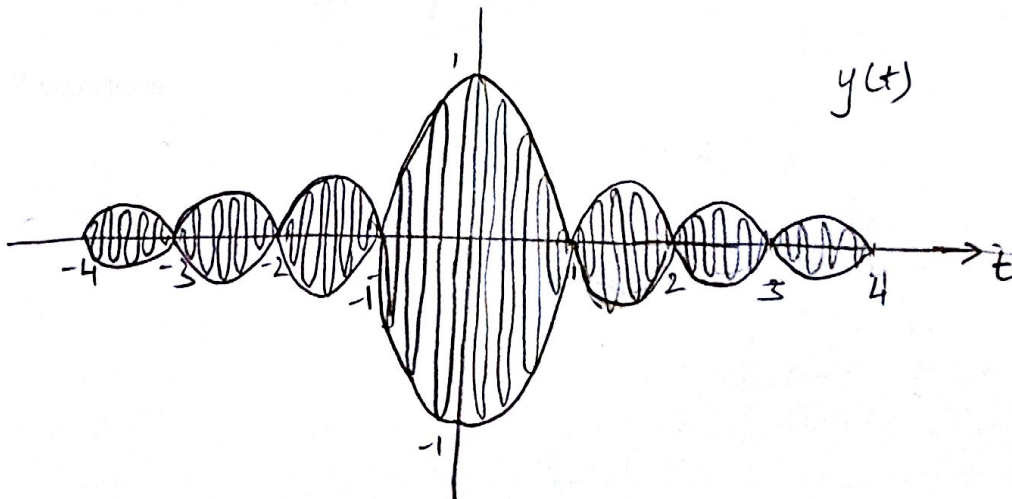
If $x_g(\omega) = u(\omega+3) - u(\omega-3)$, then $x(t) = \dfrac{\sin 3t}{\pi t}$

4. Yes, $z(t)$ & $\hat{x}(t)$ will sound identical since their spectrums are identical. Modulation & Demodulation does not introduce any content in the $0-1$ Hz frequency range & so the filtered output $\hat{x}(t)$ is identical to $z(t)$. However, $z(t)$ will have lower amplitude than $\hat{x}(t)$, so its volume will be lower.