Question 1: [20%, Work-out question, Learning Objectives 1, 2, 3]

1. [2%] What does the acronym of LTI stand for?

Consider two DT-LTI systems with impulse responses

$$h_1[n] = \begin{cases} 1 & \text{if } n = -1 \text{ or } n = 4 \\ 2 & \text{if } 0 \le n \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$h_2[n] = \begin{cases} 1 & \text{if } n = 2 \\ -1 & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases}$$

2. [3%] Is the first system causal? Is the second system causal? These are yes/no questions and there is no need to justify your answers.

3. [3%] Is the first system invertible? Is the second system invertible? These are yes/no questions and there is no need to justify your answers.

Suppose we serially concatenate two systems together. Namely, for any input $x[n]$, we first pass it through the first system and denote the (intermediate) output signal by $z[n]$. Then we pass $z[n]$ through the second system and denote the final output by $y[n]$.

4. [12%] Denote the impulse response of the above serially concatenated system by $h_{\text{serial}}[n]$. Plot $h_{\text{serial}}[n]$ for the range of $n = -2$ to 8.

   Hint: It will be MUCH easier if you can directly plot $h_{\text{serial}}[n]$ without any mathematical computation.

1. Linear Time Invariant

2. System 1 → NO System 2 → YES

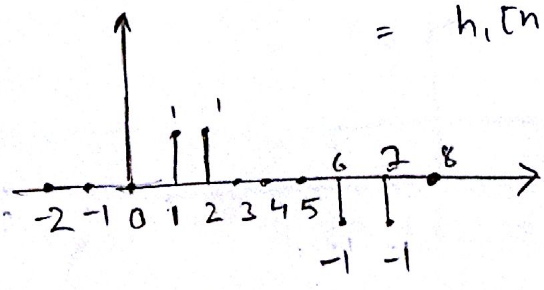
3. System 1 → NO System 2 → NO

4. $y[n] = z[n] * h_2[n] = (x[n] * h_1[n]) * h_2[n]$

$= x[n] * (h_1[n] * h_2[n])$

$\Rightarrow h_{\text{serial}}[n] = h_1[n] * h_2[n] = \sum_k h_2[k]\, h_1[n-k]$

$= h_1[n-2] - h_1[n-3]$



Question 2: [20%, Work-out question, Learning Objectives 2, 3, and 4] Consider a CT-LTI system, for which the input-output relationship is described as follows.

$$y(t) = \frac{x(t+1)}{3} + \int_{t-4}^{t+2.5} x(s)ds$$

1. [10%] Denote the impulse response of this system by $h(t)$. Compute the expression of $h(t)$ and plot it for the range of $-5 < t < 5$.

2. [10%] Compute the expression of $y(t)$ when the input being $x(t) = e^{j2t}$.

   Hint 1: You can still solve this question even if you do not know the expression of $h(t)$.

   Hint 2: You may want to consider the $\frac{x(t+1)}{3}$ term separately from the $\int_{t-4}^{t+2.5} x(s)ds$ term.

1. let $x(t) = \delta(t)$

$$\Rightarrow \quad h(t) = \frac{\delta(t+1)}{3} + \int_{t-4}^{t+2.5} \delta(s)ds$$

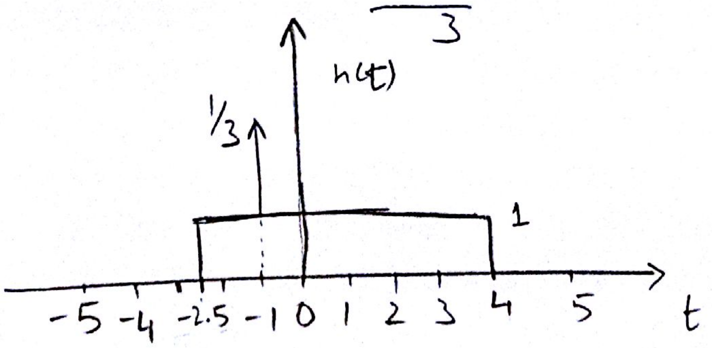
case ① $\quad t-4 > 0 \quad \Rightarrow \int_{t-4}^{t+2.5} \delta(s)ds = 0$

case ② $\quad t-4 \leq 0 \quad \& \quad t+2.5 \geq 0 \quad \Rightarrow \quad -2.5 \leq t \leq 4$

$$\Rightarrow \int_{t-4}^{t+2.5} \delta(s)\,ds = 1$$

case ③ $\quad t+2.5 < 0 \quad \Rightarrow \int_{t-4}^{t+2.5} \delta(s)ds = 0$

$$\therefore \quad h(t) = \begin{cases} \frac{\delta(t+1)}{3} + 1 & ; \ -2.5 \leq t \leq 4 \\ \frac{\delta(t+1)}{3} & \text{otherwise} \end{cases}$$



2. $H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} \frac{\delta(t+1)}{3} e^{-j\omega t} dt + \int_{-2.5}^{4} e^{-j\omega t} dt$$

$$= \frac{e^{-j\omega(-1)}}{3} + \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-2.5}^{4}$$

$$= \frac{e^{j\omega}}{3} + \left(\frac{-1}{j\omega}\right)\left(e^{-j4\omega} - e^{j\frac{5}{2}\omega}\right)$$

$x(t) = e^{j2t} = e^{j\omega_0 t}$; $\omega_0 = 2$

$$H(j\omega_0) = \frac{e^{j2}}{3} - \frac{1}{2j}\left(e^{-j8} - e^{+j5}\right)$$

$$\therefore\ y(t) = e^{j2t} H(j2)$$

$$y(t) = e^{j2t}\left[\frac{e^{j2}}{3} - \frac{1}{2j}\left(e^{-j8} - e^{+j5}\right)\right]$$

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Question 3: [10%, Work-out question, Learning Objectives 4, and 5] Consider the following signal.

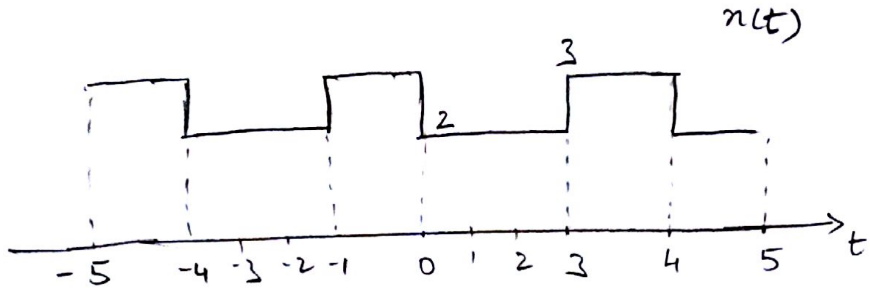
$$x(t) = \begin{cases} 2 & \text{if } 0 \leq t < 3 \\ 3 & \text{if } 3 \leq t < 4 \\ \text{periodic with period } T = 4 & \end{cases} \tag{1}$$

Plot $x(t)$ for the range of $-5 \leq t \leq 5$. And compute the Fourier series representation of $x(t)$.

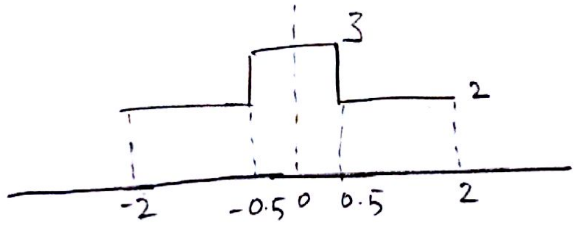
Hint: For a signal being

$$y(t) = \begin{cases} 1 & \text{if } |t| \leq T_1 \\ 0 & \text{if } T_1 < |t| < \frac{T}{2} \\ \text{periodic with period } T & \end{cases} \tag{2}$$

its Fourier series coefficients are $b_0 = \frac{2T_1}{T}$ and $b_k = \frac{\sin(k\frac{2\pi}{T}T_1)}{k\pi}$ for all $k \neq 0$.



Consider $z(t) = x(t+0.5)$ in the interval $[-2, 2]$



It is also periodic with period $T = 4$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$

In the interval $[-2, 2]$, $z(t) = \underbrace{2}_{z_1(t)} + \underbrace{y(t)}_{z_2(t)}$ with $T_1 = 0.5$ & $T = 4$

F.S. for $z_1(t)$: $a_0 = 2$, $a_k = 0 \quad \forall\, k \neq 0$

F.S. for $z_2(t)$: $b_0 = \frac{2 \times 0.5}{4} = \frac{1}{4}$, $b_k = \frac{\sin(k\pi/4)}{k\pi} \quad \forall\, k \neq 0$

F.S. for $z(t)$: $c_0 = a_0 + b_0 = 2 + \frac{1}{4} = \frac{9}{4}$

$$c_k = \frac{\sin(k\pi/4)}{k\pi} \quad \forall\, k \neq 0$$

Let F.S. of $x(t)$ be d_k

By the time shift property of F.S.

$$d_k \cdot e^{-jk(2\pi/4)(+0.5)} = c_k$$

$$\Rightarrow \quad d_k = c_k \cdot e^{+jk\pi/4}$$

$$d_0 = c_0 \times 1 = 9/4$$

$$d_k = \frac{\sin(k\pi/4)}{k\pi} e^{+jk\pi/4} \quad \forall\, k \neq 0$$

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Question 4: [30%, Work-out question, Learning Objectives 4, 5, and 6]

1. [20%] Consider the following signal

$$x[n] = \begin{cases} e^{jn} & \text{if } 100 \le n \le 199 \\ 0 & \text{if } 200 \le n \le 499 \\ \text{periodic with period } 400 & \end{cases} \quad (3)$$

   Compute the Fourier series representation of $x[n]$.

2. [10%] Consider another signal $y[n] = \sin(\frac{8\pi}{3}n)$. Compute the Fourier series representation of $y[n]$.

Hint: You may need to use the following two formulas: If $|r| \neq 1$, then we have

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}. \quad (4)$$

If $|r| < 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}. \quad (5)$$

If $|r| \geq 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} \text{ does not exist.} \quad (6)$$

1. $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{400}$

$a_0 = \frac{1}{400} \sum_{n=100}^{499} x[n] = \frac{1}{400} \sum_{n=100}^{199} e^{jn}$

Let $l = n - 99$, $\quad a_0 = \frac{1}{400} \sum_{l=1}^{100} e^{j(l+99)} = \frac{e^{j100}}{400} \sum_{l=1}^{100} e^{j(l-1)}$

$a_0 = \frac{e^{j100}}{400} \times \frac{1 - e^{j100}}{1 - e^{j}}$

$a_k = \frac{1}{400} \sum_{n=100}^{199} e^{jn} e^{-jk\frac{2\pi}{400}n}$, let $l = n - 99$

$= \frac{1}{400} \sum_{l=1}^{100} e^{j(l+99)} e^{-jk\frac{2\pi}{400}(l+99)}$

$$a_K = \frac{1}{400} \times e^{j100} \times e^{-jK\frac{2\pi}{400}100} \times \sum_{l=1}^{100} e^{j(l-1)} e^{-j\frac{K2\pi}{400}(l-1)}$$

$$= \frac{e^{j100} \times e^{-jK\pi/2}}{400} \sum_{l=1}^{100} \left[e^{j\left[1 - \frac{K2\pi}{400}\right]} \right]^{(l-1)}$$

$$= \frac{e^{j100}(-j)^K}{400} \times \frac{1 - e^{j\left(1-\frac{K2\pi}{400}\right)100}}{1 - e^{j\left(1-\frac{K2\pi}{400}\right)}}$$

$$a_K = \frac{e^{j100}(-j)^k}{400} \times \frac{1 - e^{j100}(-j)^K}{1 - e^{j} e^{-j\frac{K2\pi}{400}}} \qquad \forall\, K \neq 0$$

$K = 1, 2, \ldots, 399$

2. $y[n] = \sin\left(\frac{8\pi}{3} n\right)$ period $= \frac{2\pi}{8\pi/3} = \frac{3}{4}$ → must be integer

$\Rightarrow N = 3$

$$= \sin\left(\left(2\pi + \frac{2\pi}{3}\right) n\right)$$

$$= \sin\left(\frac{2\pi}{3} n\right) \qquad \Rightarrow w_0 = \frac{2\pi}{N} = \frac{2\pi}{3}$$

$$= \frac{e^{j2\pi/3\, n} - e^{-j2\pi/3\, n}}{2j}$$

$$\Rightarrow a_1 = \frac{1}{2j}, \quad a_{-1} = a_{3-1} = a_2 = \frac{-1}{2j}$$

$$\therefore a_0 = 0, \quad a_1 = \frac{1}{2j}, \quad a_2 = \frac{-1}{2j}$$

Question 5: [20%, Multiple Choices] The following questions are multiple-choice questions and there is no need to justify your answers. Consider the following two systems:

System 1: When the input is $x_1(t)$, the output is

$$y_1(t) = \begin{cases} \int_{2t-1}^{2t} x_1(2s)ds & \text{if } t \leq 0 \\ \int_{-\infty}^{t} x_1(s)ds & \text{if } t > 0 \end{cases} \tag{7}$$

System 2: When the input is $x_2[n]$, the output is

$$y_2[n] = \sum_{k=0}^{|x_2[n]|} x_2[n]e^{j2\pi k} \tag{8}$$

Answer the following questions

1. [4%, Learning Objective 1] Is System 1 memoryless? Is System 2 memoryless?
2. [4%, Learning Objective 1] Is System 1 causal? Is System 2 causal?
3. [4%, Learning Objective 1] Is System 1 stable? Is System 2 stable?
4. [4%, Learning Objective 1] Is System 1 linear? Is System 2 linear?
5. [4%, Learning Objective 1] Is System 1 time-invariant? Is System 2 time-invariant?

SYSTEM 1	SYSTEM 2
With Memory	Memoryless
Causal	Causal
Unstable	Stable
Linear	NOT Linear
Not Time Invariant	Time Invariant