Question 1: [20%, Work-out question, Learning Objectives 2 and 3] Consider two continuous-time signals $x(t)$ and $y(t)$.

$$x(t) = \begin{cases} e^{-t} & \text{if } 2 < t \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = \begin{cases} \cos(2t)e^{-t} & \text{if } 1 < t \\ 0 & \text{otherwise} \end{cases}$$

Compute the expression of

$$z(t) = \int_{s=-\infty}^{\infty} x(t-s)y(s)ds.$$

Hint 1: You may need to use the following formulas:

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{1}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}. \tag{2}$$

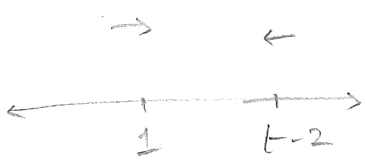
Hint 2: You need to study different cases in your answer. The computation of some cases may be significantly longer and you may want to work on some other questions first before you come back to finish the computation of some of the cases.

$$x(t-s) = \begin{cases} e^{-(t-s)}, & \text{if } (t-s) > 2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} e^{-(t-s)} & \text{if } s < t-2 \\ 0 & \text{otherwise} \end{cases}$$

$$y(s) = \begin{cases} \cos(2s)\, e^{-s}, & \text{if } s > 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{e^{j2s} + e^{-j2s}}{2} \cdot e^{-s}, & \text{if } s > 1 \\ 0, & \text{otherwise} \end{cases}$$

The integral $z(t)$ is non-zero when the product $x(t-s)\,y(s) \neq 0$

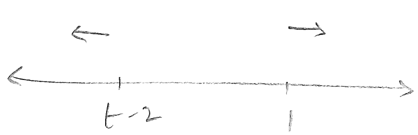
(I)



$t-2 > 1 \Rightarrow t > 3$

limits $\rightarrow$ 1 to $t-2$

(II)



$t-2 < 1 \Rightarrow t < 3$

No common domain over which product is non-zero

$\Rightarrow$ Integral 0 for $t < 3$

$\Rightarrow z(t) = 0$ if $t < 3$

for $t > 3$

$$z(t) = \int_{1}^{t-2} e^{-(t-s)} \cdot \frac{e^{j2s} + e^{-j2s}}{2} \cdot e^{-s}\, ds$$

$$= e^{-t} \cdot \int_{1}^{t-2} \cos 2s \, ds$$

$$= e^{-t} \cdot \left. \frac{\sin 2s}{2} \right|_{1}^{t-2}$$

$$= \frac{e^{-t}}{2} \left\{ \sin(2(t-2)) - \sin(2) \right\}$$

$$= \frac{e^{-t}}{2} \cdot \left[\sin(2t-4) - \sin(2) \right]$$

$$\Rightarrow \quad z(t) = \begin{cases} \frac{e^{-t}}{2} \cdot \left[\sin(2t-4) - \sin(2) \right], & \text{if } t > 3 \\ 0, & \text{if } t < 3 \end{cases}$$

Question 2: [15%, Work-out question, Learning Objectives 1, 4, and 5] Given a discrete-time signal

$$x[n] = \begin{cases} 2^{n-10}e^{jn} & \text{if } 15 \le n \\ 0 & \text{otherwise} \end{cases}$$

Based on $x[n]$, we construct another function

$$y(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}. \tag{3}$$

1. [7.5%] Compute the average power $x[n]$ for the duration of $n = -100$ to 100.
2. [7.5%] Compute the value of $y(3+4j)$.

Hint: You may need to use the following two formulas: If $|r| \neq 1$, then we have

$$\sum_{k=1}^{K} ar^{k-1} = \frac{a(1-r^K)}{1-r}. \tag{4}$$

If $|r| < 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}. \tag{5}$$

If $|r| \geq 1$, then we have

$$\sum_{k=1}^{\infty} ar^{k-1} \text{ does not exist.} \tag{6}$$

① $P_{av} = \frac{1}{100-(-100)+1} \sum_{n=-100}^{100} |x[n]|^2 = \frac{1}{201} \sum_{n=15}^{100} |2^{n-10} e^{jn}|^2$

$= \frac{1}{201} \sum_{n=15}^{100} 2^{2n} \cdot 2^{-20} \cdot \underbrace{|\cancel{e^{jn}}|^2}_{1} = \frac{1}{201} \sum_{n=15}^{100} 4^n \cdot 2^{-20}$

$= \frac{2^{-20}}{201} \sum_{n=15}^{100} 4^n$ let $m = n-14$, sub. in sum $\Rightarrow n = m+14$

$= \frac{2^{-20}}{201} \sum_{m=1}^{86} 4^m \cdot 4^{14} = \frac{4 \cdot 2^{-20} \cdot 2^{28}}{201} \sum_{m=1}^{86} 1 \cdot (4)^{m-1}$

$$= \frac{2^{10}}{201} \times \frac{1.(1-4^{86})}{(1-4)} = \frac{2^{10}.(4^{86}-1)}{603}$$

(2) $$y(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$z = 3+4j = 5\left(\frac{3}{5}+\frac{4}{5}j\right) = 5(\cos\theta + j\sin\theta)$$

where $\theta = \sin^{-1}(4/5) \approx 53^\circ = 53 \times \frac{\pi}{180}$ rad

$$\Rightarrow \quad z = 5e^{j\theta}$$

$$y(5e^{j\theta}) = \sum_{n=15}^{\infty} 2^{n-10}.e^{jn}.5^{-n}e^{-j\theta n}$$

$$= 2^{-10}\sum_{n=15}^{\infty}\left(\frac{2}{5}\right)^n e^{j(1-\theta)n}.$$

$$= 2^{-10}\sum_{n=15}^{\infty}\left(\frac{2}{5}e^{j(1-\theta)}\right)^n$$

let $m = n-14$, $\Rightarrow n = m+14$

$$y(5e^{j\theta}) = 2^{-10}\left\{\sum_{m=1}^{\infty}\left(\frac{2}{5}e^{j(1-\theta)}\right)^{m-1}\right\}\left(\frac{2}{5}\right)^{15} e^{j(1-\theta).15}$$

$$= 5^{-15}.2^{5}.e^{j(1-\theta).15} . \sum_{m=1}^{\infty} 1.\, r^{m-1}$$

where $r = \frac{2}{5}.e^{j(1-\theta)}$

$$|r| = \left|\frac{2}{5}.e^{j(1-\theta)}\right| = \left|\frac{2}{5}\right|.\left|e^{j(1-\theta)}\right| = \frac{2}{5} < 1$$

$$y(3+4j) = \frac{2^5.e^{j15}}{(3+4j)^{15}} \times \frac{1}{1-r} = \frac{32e^{j15}}{(3+4j)^{15}} \times \frac{1}{1-\frac{2e^{j}}{3+4j}}$$

$$= \frac{32e^{j15}}{(3+4j)^{15}} \times \frac{(3+4j)}{3+4j-2e^{j}} = \frac{32e^{j15}}{(3+4j)^{14}(3+4j-2e^{j})}$$

Question 3: [16%, Work-out question, Learning Objectives 1, 4, and 6] Consider the following discrete time signal.

$$x[n] = e^{j\frac{5\pi}{2}n} + e^{-j\pi n}.$$

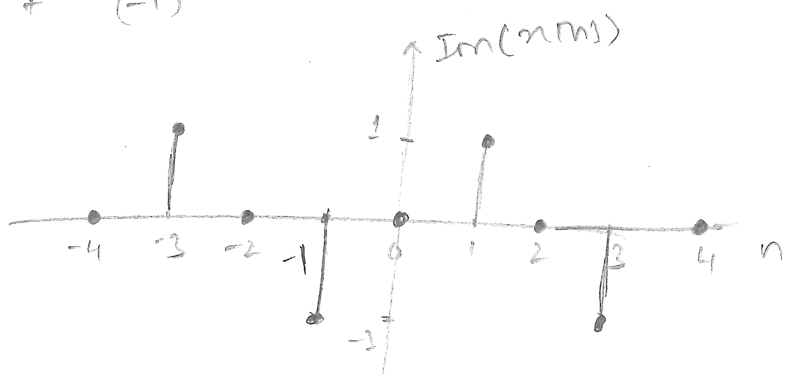
1. [6%] Plot the imaginary part of $x[n]$ for the range of $-4 \leq n \leq 4$.
2. [6%] Find out the fundamental period of $x[n]$.
3. [4%] Find out the instantaneous power of $x[n]$ when $n = 3$.

Hint: You may need the formula that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.

$$x[n] = \left(\cos\frac{5\pi}{2}n + j\sin\frac{5\pi}{2}n\right) + \left(\cos(-\pi n) + j\,\cancel{\sin(-\pi n)}^{\,0}\right)$$

$$= \left(\cos\frac{\pi}{2}n + j\sin\frac{\pi}{2}n\right) + (-1)^n$$

① $Im(x[n]) = \sin\frac{\pi}{2}n$



② $x[n] = \cos\frac{\pi}{2}n + \cos\pi n + j\sin\frac{\pi}{2}n$

$N_1 = \frac{2\pi}{\pi/2} = 4 \qquad N_2 = \frac{2\pi}{\pi} = 2 \qquad N_3 = \frac{2\pi}{\pi/2} = 4$

Fundamental period $= LCM(N_1, N_2, N_3) = \underline{\underline{4}}$

③ $P_{inst.} = |x[n]|^2$

$x[3] = e^{j\frac{5\pi}{2}\cdot 3} + e^{-j\pi\cdot 3}$

$= \cos\frac{3\pi}{2} + \cos 3\pi + j\sin\frac{3\pi}{2}$

$= 0 + -1 + j(-1)$

$= -1 - j$

$P_{inst} = |-1-j|^2 = 1^2 + 1^2 = \underline{\underline{2}}$

Question 4: [15%, Work-out question, Learning Objectives 1, 2, and 6] Consider the following signal.

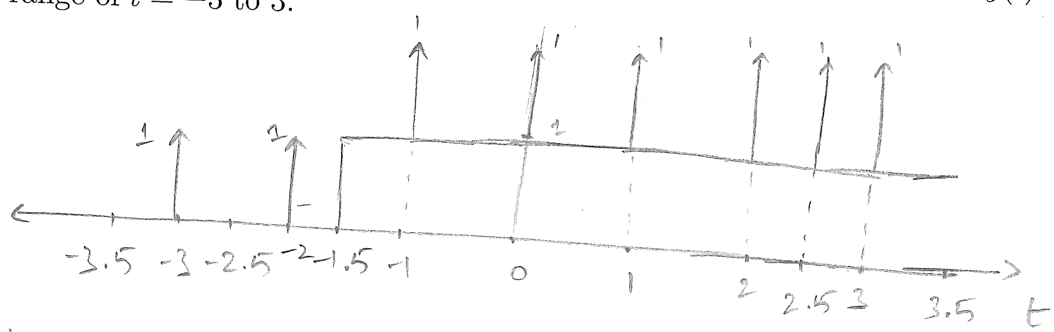
$$x(t) = \delta(t - 2.5) + \left(\sum_{k=-\infty}^{\infty} \delta(t - k) \right) + U(t + 1.5) \qquad (7)$$

1. [7%] Plot $x(t)$ for the range of $t = -3.5$ to 3.5.
2. [8%] Plot the odd part of $x(t)$ for the range of $t = -3.5$ to 3.5.

   Hint: For the second sub-question, if you do not know how to plot $x(t)$ in the first sub-question, you can solve the following alternative question instead. You will still get 5 points if your answer is correct.

   Alternative question: Suppose $y(t) = e^t U(t - 1)$. Plot the odd part of $y(t)$ for the range of $t = -3$ to 3.

①

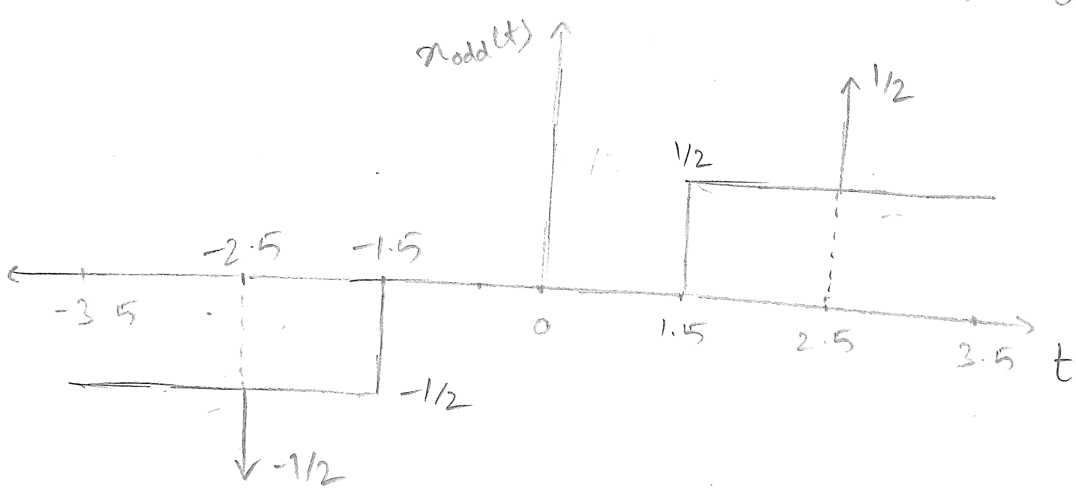


②

$$x_{odd}(t) = \frac{x(t) - x(-t)}{2} = \frac{1}{2}\Big\{ \delta(t-2.5) - \delta(-t-2.5) + \sum_{k=-\infty}^{\infty} \delta(t-k) - \delta(-t-k) + U(t+1.5) - U(-t+1.5) \Big\}$$

$$\sum_{k=-\infty}^{\infty} \delta(t-k) - \delta(-t-k) = 0$$ since $\delta(t-k_1) = \delta(t+k_1)$ from another term of sum & $\delta(-t-k_1) = \delta(t+k_1)$

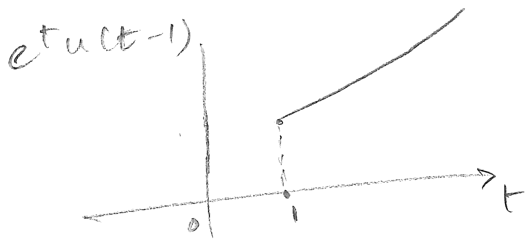
$$x_{odd}(t) = \frac{1}{2}\Big\{ \delta(t-2.5) - \delta(-t-2.5) + U(t+1.5) - U(-t+1.5) \Big\}$$

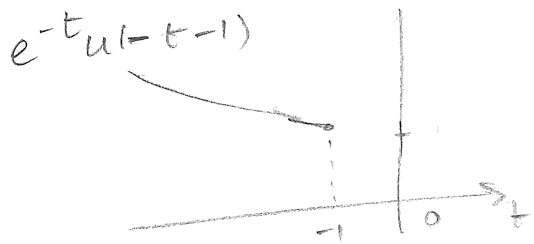


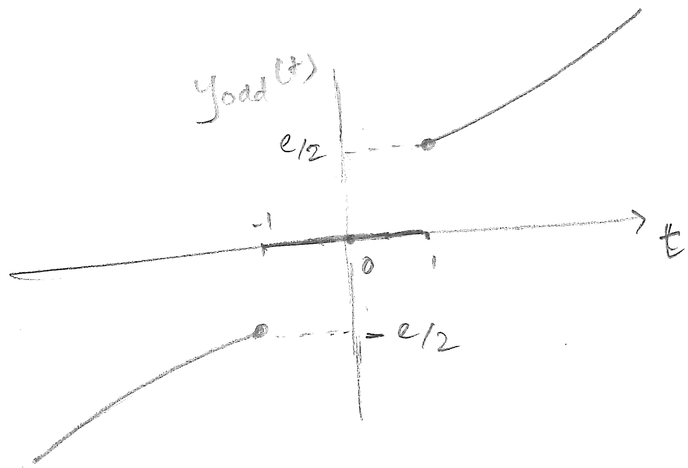
Alt. Ans.

$$y(t) = e^{t} u(t-1)$$

$$y_{odd}(t) = \frac{y(t) - y(-t)}{2} = \frac{e^{t} u(t-1) - e^{-t} u(-t-1)}{2}$$





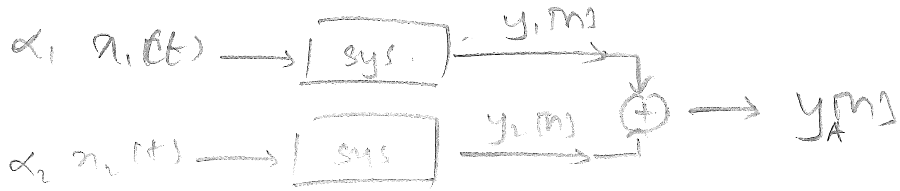


Question 5: [14%, Work-out question, Learning Objective 1]

Consider the following system that takes a continuous-time signal $x(t)$ as input and outputs a discrete-time signal $y[n]$:

$$y[n] = x(0.25n^2) - 3x(-n^3) \tag{8}$$

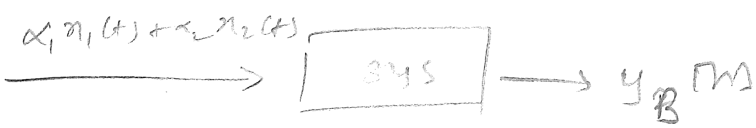
Is the above system is linear or not? Carefully explain the steps how you decide whether the system is linear or not.



$y_1[n] = \alpha_1 x_1(0.25n^2) - 3\alpha_1 x_1(-n^3)$

$y_2[n] = \alpha_2 x_2(0.25n^2) - 3\alpha_2 x_2(-n^3)$

$y_A[n] = \alpha_1 x_1(0.25n^2) + \alpha_2 x_2(0.25n^2) - 3\alpha_1 x_1(-n^3)$
$\quad - 3\alpha_2 x_2(-n^3)$



If $z(t) = a(t) + b(t)$ then $c[n] = z(n^2) = a(n^2) + b(n^2)$

similarly to above

$y_B[n] = \alpha_1 x_1(0.25n^2) + \alpha_2 x_2(0.25n^2)$
$\quad - 3\alpha_1 x_1(-n^3) - 3\alpha_2 x_2(-n^3)$

$\because$ $y_A[n] = y_B[n]$, system is linear.

Question 6: [20%, Multiple Choices] The following questions are multiple-choice questions and there is no need to justify your answers. Consider two continuous-time signals:

$$x_1(t) = e^{2\sin(t)} \cdot e^{-2\sin(\pi t)}$$
$$x_2(t) = e^{jt^2-|t|}(1+j)^{t^4} = \sqrt{2}\, e^{jt^2-|t|}\, e^{j\pi/4\, t^4}$$

and two discrete-time signals:

$$x_3[n] = \cos(\pi n^2) + \sin(\frac{\pi}{3}n)$$
$$x_4[n] = \cos(\pi n)U[n-1] - \cos(3\pi n)U[n-2] + \delta[n+1].$$

1. [10%, Learning Objective 1] For $x_1(t)$ to $x_4[n]$, determine whether it is periodic or not, *respectively.* If it is periodic, write down the fundamental period. Please state explicitly which signal is periodic and which is not.

2. [10%, Learning Objective 1] For $x_1(t)$ to $x_4[n]$, determine whether it is even or odd or neither of them, respectively. Please state explicitly which signal is even, which is odd, and which is neither.

① $x_1(t) \to$ not periodic
$x_2(t) \to$ not periodic
$x_3[n] \to$ periodic, $N = 6$
$x_4[n] \to$ not periodic

② $x_1(t) \to$ neither
$x_2(t) \to$ even
$x_3[n] \to$ neither
$x_4[n] \to$ odd